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written/compiled by top Educators

creative, interactive, concise approach
MATHS GRADE 12 NCAPS
TEXTBOOK INFORMATION

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The CONCEPT

The purpose of this publication is to cover all Maths topics in the new curriculum (CAPS) using a user-friendly, modern approach. The examples presented in each chapter of the textbook cover all of the main concepts in each topic. There is a logical, to-the-point, progression from one example to the next and the exercises reinforce the concepts inherent in the particular topic dealt with.

At the end of each chapter there is a mixed revision exercise. This exercise is for revising all of the concepts dealt with in the chapter. There is also a “Some Challenges” exercise which provides invaluable extension and problem solving for top learners. At the end of the book, short answers to all of the exercises have been provided.

ENDORSEMENT

“This textbook has always been a life-saver for me. I just don’t have the time to waste trying to create my own lessons using other books and then still trying to get to assess my learners in the way that we are supposed to. This textbook has helped me to get through the content as quickly and effectively as possible leaving more time for me to assess my learners. I can also have a life outside of school and not get so wrapped up with so much work. I like the assessment tasks in the Teacher Guide. I have used them to meet the requirements in the policy documents.”

Hester Jansen Van Vuuren, Educator
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CHAPTER 1 – SEQUENCES AND SERIES

In Grade 10 and 11 you learnt about linear and quadratic number patterns. Linear number patterns have a constant difference between consecutive terms while quadratic number patterns have a constant second difference.

REVISION OF QUADRATIC NUMBER PATTERNS

In Grade 11 we dealt with quadratic number patterns having a general term of the form \( T_n = an^2 + bn + c \).

EXAMPLE

Consider the following number pattern: 2; 6; 11; ...

(a) Determine the \( n \)th term (general term) and hence the value of the 42nd term.

(b) Determine which term will equal 1091.

Solutions

(a) \[
\begin{align*}
2a &= 2 \\
3a + b &= 1 \\
a + b + c &= 2
\end{align*}
\]

\[
\therefore a = 1 \\
\therefore 3(l) + b = 1 \\
\therefore (l) + (-2) = 2
\]

\[
\therefore b = -2 \\
\therefore c = 3
\]

\[
\therefore T_n = n^2 - 2n + 3
\]

\[
\therefore T_{42} = (42)^2 - 2(42) + 3 = 1683
\]

(b) \( T_n = 1091 \)

\[
\therefore n^2 - 2n + 3 = 1091
\]

\[
\therefore n^2 - 2n - 1088 = 0
\]

\[
\therefore (n - 34)(n + 32) = 0
\]

\[
\therefore n = 34 \text{ or } n = -32
\]

But \( n \neq -32 \)

\[
\therefore n = 34
\]

The 34th term will equal 1091

REVISION EXERCISE

1. Determine the general term for each number pattern below.

(a) 2; 6; 14; 26; ... 
(b) 4; 9; 16; 25; ...
(c) 1; 3; 6; 10; ...
(d) -1; 0; 3; 8; ...
(e) -3; -6; -11; -18; ...
(f) 10; 6; 3; 1; ...

2. Consider the following number pattern: \(1; 6; 15; 28; \ldots\)
   
   (a) Determine the general term and hence the value of the 20th term.
   
   (b) Determine which term will equal 3160.

3. Consider the following number pattern: \(-4; -10; -18; -28; \ldots\)
   
   (a) Determine the general term and hence the value of the 25th term.
   
   (b) Determine which term will equal \(-810\).

4. Consider the following pattern that emerges when you add the terms of a linear number pattern.

First the linear number pattern: \(3; 7; 11; 15; \ldots\)

<table>
<thead>
<tr>
<th>Line 1</th>
<th>Line 2</th>
<th>Line 3</th>
<th>Line 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3 + 7 = 10</td>
<td>3 + 7 + 11 = 21</td>
<td>3 + 7 + 11 + 15 = 36</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a) Determine the sum of the numbers in Line 5 and 6.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b) What type of number pattern is formed by the sum of the numbers in each line?</td>
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<td></td>
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<tr>
<td>(c) Hence or otherwise determine the sum of the numbers in Line (n).</td>
<td></td>
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</tbody>
</table>

**ARITHMETIC SEQUENCES**

Consider the following linear number pattern: \(7; 10; 13; 16; 19; \ldots\)

We can rewrite this pattern using only the first term and the constant difference.

\(T_1 = 7\) and the constant or common difference = \(T_2 - T_1 = T_3 - T_2 = T_4 - T_3 = \ldots = 3\)

\(T_2 = 7 + 3\)

\(T_3 = 7 + 3 + 3 = 7 + 2(3)\)

\(T_4 = 7 + 3 + 3 + 3 = 7 + 3(3)\)

\(T_5 = 7 + 3 + 3 + 3 + 3 = 7 + 4(3)\)

\(T_{10} = 7 + 9(3)\)

\(T_{100} = 7 + 99(3)\)

\(\therefore T_n = 7 + (n - 1)(3)\)

Therefore the general term of the pattern is: \(T_n = 7 + (n - 1)(3)\)

The general term can be simplified further as follows:

\(T_n = 7 + (n - 1)(3)\)

\(\therefore T_n = 7 + 3n - 3\)

\(\therefore T_n = 4 + 3n\)

If the letter \(a\) is used for the first term and \(d\) for the constant difference of a linear pattern, then the pattern can be written as follows:

\(T_1 = a\)

\(T_2 = a + d\)

\(T_3 = (a + d) + d = a + 2d\)

\(T_4 = (a + 2d) + d = a + 3d\)
\[ T_3 = (a + 3d) + d = a + 4d \]

\[ T_6 = a + 5d \]

\[ T_{10} = a + 9d \]

\[ T_{100} = a + 99d \]

\[ \therefore T_n = a + (n-1)d \]

Therefore the general term of the pattern is: \( T_n = a + (n-1)d \)

**Linear patterns** are also called arithmetic sequences and have a general term of

\[ T_n = a + (n-1)d \]

where:

- \( a \) represents the first term
- \( d \) represents the constant or common difference
- \( n \) represents the position of the term
- \( T_n \) represents the \( n \)th term or general term (the value of the term in the \( n \)th position)

**EXAMPLE 1**

Consider the arithmetic sequence \( 3; 5; 7; 9; \ldots \)

(a) Determine a formula for the general term of the above sequence.

(b) Find the value of the 50\(^{th}\) term.

**Solution**

(a) \[ T_1 = 3 \]

\[ \therefore a = 3 \]

\[ T_2 - T_1 = 5 - 3 = 2 \]

\[ T_3 - T_2 = 7 - 5 = 2 \]

\[ \therefore d = 2 \]

To find the general term \( T_n \) you have to substitute \( a = 3 \) and \( d = 2 \) into the general term for an arithmetic sequence, namely \( T_n = a + (n-1)d \)

\[ \therefore T_n = 3 + (n-1)(2) \]

\[ \therefore T_n = 3 + 2n - 2 \]

\[ \therefore T_n = 2n + 1 \]

(b) \[ T_n = 2n + 1 \]

\[ \therefore T_{50} = 2(50) + 1 = 101 \]

**EXAMPLE 2**

Consider the following sequence \( -5; -9; -13; -17; \ldots \)

(a) Show that the sequence is arithmetic.

(b) Find the value of the 25\(^{th}\) term of the sequence.

**Solutions**

(a) \[ T_2 - T_1 = -9 - (-5) = -4 \]

\[ T_2 - T_1 = -13 - (-9) = -4 \]

There is a constant difference of \(-4\)

\[ \therefore T_{25} = -5 + (25 - 1)(-4) = -101 \]

\[ T_n = a + (n-1)d \]

\[ a = -5, \ d = -4 \ \text{and} \ n = 25 \]
EXAMPLE 3

Determine which term of the sequence \(-1; 2; 5; 8; \ldots\) is equal to 80.

Solution

\[T_2 - T_1 = 2 - (-1) = 3\] and \[T_3 - T_2 = 5 - 2 = 3\]

\[\therefore d = 3\] (the sequence is arithmetic)

Write down what is given:

\[a = -1, \quad d = 3\] and \[T_n = 80\] (we know the actual term's value but not the position or \(n\) value)

\[T_n = a + (n - 1)d\] (state the formula)

\[\therefore 80 = -1 + (n - 1)(3)\] (substitute \(a = -1, \quad d = 3\) and \(T_n = 80\) )

\[\therefore 80 = -1 + 3n - 3\]

\[\therefore 84 = 3n\]

\[\therefore n = 28\]

\[\therefore T_{28} = 80\]

EXAMPLE 4

\(x; 4x + 5; 10x - 5\) are the first three terms of an arithmetic sequence.

Determine the value of \(x\) and hence the sequence.

Solution

Since the sequence is arithmetic, it is clear that

\[d = T_2 - T_1 = (4x + 5) - (x) = 3x - 5\]

\[d = T_3 - T_2 = (10x - 5) - (4x + 5) = 10x - 5 - 4x - 5 = 6x - 10\]

\[\therefore 3x + 5 = 6x - 10\]

\[\therefore -3x = -15\]

\[\therefore x = 5\]

\[\therefore T_1 = x = 5\]

\[\therefore T_2 = 4x + 5 = 4(5) + 5 = 25\]

\[\therefore T_3 = 10x - 5 = 10(5) - 5 = 45\]

\[\therefore\] The sequence is \(5; 25; 45; \ldots\)

EXERCISE 1

1. Determine the general term of the following arithmetic sequences:
   (a) \(-1; 3; 7; \ldots\)  (b) \(4; -2; -8; \ldots\)  (c) \(1; -1; -3; \ldots\)
   (d) \(99; 106; 113; \ldots\)

2. Determine the 38th term for each of the following arithmetic sequences:
   (a) \(-4; -8; -12; \ldots\)  (b) \(2; -1.5; -5; \ldots\)  (c) \(99; 88; 77; \ldots\)
   (d) \(6; \frac{21}{4}; \frac{9}{2}; \ldots\)  (e) \(T_k = 3k - 4\)  (f) \(T_k = -2k + 5\)
3. (a) Which term of the arithmetic sequence \(-5; -2; 1; ...\) is equal to 94?
   (b) Which term of the arithmetic sequence \(4; 2.5; 1; ...\) is equal to \(-66.5\)?
   (c) Find the number of terms in the sequence \(12; 7; 2; ...; -203\).
   (d) Find the number of terms in the sequence \(-55; -48; -41; ...; 85\).

4. Given the arithmetic sequence: \(-29; -23; -17; ...\)
   (a) Determine the 31st term.
   (b) Determine which term is equal to 31.

5. Given the arithmetic sequence: \(-13; -9; -5; ...\)
   (a) Which term in the above sequence is 51?
   (b) Calculate the 51st term.

6. \(p; 2p + 2; 5p + 3; ...\) are the first three terms of an arithmetic sequence.
   (a) Calculate the value of \(p\).
   (b) Determine the sequence.
   (c) Find the 49th term.
   (d) Which term of the sequence is \(100\frac{1}{2}\)?

7. \(x + 3; 2x + 6; 3x + 9; ...\) are the first three terms of an arithmetic sequence.
   (a) Determine the 10th term in terms of \(x\).
   (b) Determine the \(n\)th term in terms of \(x\).

**GEOMETRIC SEQUENCES**

Consider the following number pattern: \(6; 12; 24; 48; 96; ...\)
This pattern is not linear (arithmetic) since there is no constant difference between the terms. In this pattern each successive term is obtained by multiplying the previous term by 2.
Notice too that the following ratios between the terms are also equal to 2:
\[
\frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3} = ... = 2
\]
There is a **constant ratio** for this number pattern and this kind of number pattern is called an **exponential** or **geometric sequence**.
We can rewrite this pattern using only the first term and the constant ratio:
\[
T_1 = 6
\]
\[
T_2 = 6 \times 2
\]
\[
T_3 = 6 \times 2 \times 2 = 6 \times 2^2
\]
\[
T_4 = 6 \times 2 \times 2 \times 2 = 6 \times 2^3
\]
\[
T_5 = 6 \times 2^4
\]
\[
T_{10} = 6 \times 2^9
\]
\[
T_{100} = 6 \times 2^{99}
\]
\[
\therefore T_n = 6 \times 2^{n-1}
\]
Therefore the general term of the pattern is: \(T_n = 6 \times 2^{n-1}\)

If the letter \(a\) is used for the first term and \(r\) for the constant ratio of an exponential number pattern, then the pattern can be written as follows:
\[ T_1 = a \]
\[ T_2 = a \times r = ar \]
\[ T_3 = (a \times r) \times r = ar^2 \]
\[ T_4 = (a \times r^2) \times r = ar^3 \]
\[ T_5 = ar^4 \]
\[ T_{10} = ar^9 \]
\[ T_{100} = ar^{99} \]
\[ \therefore T_n = ar^{n-1} \]

Therefore the general term of the pattern is: \( T_n = ar^{n-1} \)

**Exponential or geometric number patterns** have a general term: \( T_n = ar^{n-1} \)

where:
- \( a \) represents the first term
- \( d \) represents the constant or common ratio
- \( n \) represents the position of the term
- \( T_n \) represents the \( n \)th term or general term (the value of the term in the \( n \)th position)

**EXAMPLE 5**

Consider the geometric sequence \( 2; 3; 4,5; 6,25; ... \)

(a) Determine a formula for the general term of the sequence.

(b) Find the value of the 15th term.

**Solutions**

(a) \( T_1 = a = 2 \)

\[ \frac{T_2}{T_1} = \frac{3}{2} \quad \text{and} \quad \frac{T_3}{T_2} = \frac{4.5}{3} = \frac{3}{2} \]

\[ \therefore r = \frac{3}{2} \]

Substitute \( a = 2 \) and \( r = \frac{3}{2} \) into the general term \( T_n = ar^{n-1} \)

\[ T_n = 2 \left( \frac{3}{2} \right)^{n-1} \]

(b) \( T_n = 2 \left( \frac{3}{2} \right)^{n-1} \)

\[ T_{15} = 2 \left( \frac{3}{2} \right)^{15-1} \]

\[ \therefore T_{15} = 2 \left( \frac{3}{2} \right)^{14} \]

\[ \therefore T_{15} = 583.86 \text{ (rounded off to two decimal places)} \]
EXAMPLE 6

Consider the following sequence –5; 10; –20; 40;...
(a) Show that the above sequence is geometric.
(b) Determine the value of the 20th term of the above sequence.

Solution

(a) \( \frac{T_2}{T_1} = \frac{10}{-5} = -2 \) and \( \frac{T_3}{T_2} = \frac{-20}{10} = -2 \) (constant ratio = -2)

(b) \( T_n = ar^{n-1} \) with \( a = -5 \), \( r = -2 \) and \( n = 20 \)
   \[ \therefore T_{20} = (-5)(-2)^{20-1} = (-5)(-2)^{19} = 2 \, 621 \, 440 \]

Revision of basic exponential equations and other types

Exponential equations and equations involving odd or even exponents form an integral part of the examples and exercises which follow in this chapter. It is therefore advisable to revise these basic equations before proceeding.

Consider the following examples of exponential equations:

Solve for \( n \):

(a) \( 64 = 2^n \)
   \[ \therefore 2^6 = 2^n \]
   \[ \therefore 6 = n \]
   \[ \therefore n = 6 \]

(b) \( 64 = (-2)^n \)
   \[ \therefore (-2)^6 = (-2)^n \]
   \[ \therefore 6 = n \]
   \[ \therefore n = 6 \]

(c) \( \frac{1}{81} = \left(\frac{1}{3}\right)^n \)
   \[ \therefore \left(\frac{1}{3}\right)^4 = \left(\frac{1}{3}\right)^n \]
   \[ \therefore 4 = n \]
   \[ \therefore n = 4 \]

(d) \( \frac{1}{128} = 2\left(-\frac{1}{2}\right)^n \)
   \[ \therefore \frac{1}{128} \times \frac{1}{2} = 2\left(-\frac{1}{2}\right)^n \times \frac{1}{2} \]
   \[ \therefore \frac{1}{256} = \left(-\frac{1}{2}\right)^n \]
   \[ \therefore \left(-\frac{1}{2}\right)^8 = \left(-\frac{1}{2}\right)^n \]
   \[ \therefore n = 8 \]
Consider the following equations involving odd or even exponents:

(a) \( r^2 = 9 \)
This equation can be solved by taking the square root on both sides.
\[ \therefore r = \pm \sqrt{9} \quad (\text{Don’t forget to include } \pm \text{ since there are two solutions}) \]
\[ \therefore r = \pm 3 \]

(b) \( r^3 = 27 \)
In this equation, there is no need to include \( \pm \) since there is only one solution. Simplify take the cube root on both sides:
\[ \therefore r = \sqrt[3]{27} \]
\[ \therefore r = 3 \]

(c) \( r^3 = -27 \)
\[ \therefore r = \sqrt[3]{-27} \]
\[ \therefore r = -3 \]

(d) \( r^4 = 16 \)
This equation can be solved by taking the fourth root on both sides.
\[ \therefore r = \sqrt[4]{16} \quad (\text{Don’t forget to include } \pm \text{ since there are two solutions}) \]
\[ \therefore r = \pm 2 \]

**Summary**

Consider the equation \( r^n = a \)

If \( n \) is odd, then there is one solution: \( r = \sqrt[n]{a} \)

If \( n \) is even, then there are two solutions: \( r = \pm \sqrt[n]{a} \) provided \( a > 0 \)

**EXAMPLE 7**

Determine which term in the sequence \( 4; 12; 36; 108; \ldots \) is equal to 8748.

**Solution**

\[ \frac{T_2}{T_1} = \frac{12}{4} = 3 \quad \text{and} \quad \frac{T_3}{T_2} = \frac{36}{12} = 3 \]
\[ \therefore r = 3 \quad (\text{the sequence is geometric}) \]

\( a = 4, \ r = 3 \) and \( T_n = 8748 \) (we know the actual term's value but not the position or \( n \) value)

\[ T_n = ar^{n-1} \quad (\text{state the formula}) \]
\[ \therefore 8748 = 4(3)^{n-1} \quad (\text{substitute } a = 4, \ r = 3 \text{ and } T_n = 8748) \]
\[ \therefore \frac{8748}{4} = (3)^{n-1} \]
\[ \therefore 2187 = (3)^{n-1} \]
\[ \therefore 3^7 = (3)^{n-1} \]
\[ \therefore 7 = n - 1 \]
\[ \therefore n = 8 \]
\[ \therefore T_8 = 8748 \]
EXAMPLE 8

\( k + 1 ; k - 1 ; 2k - 5 \) are the first three terms of a geometric sequence.

Calculate the value of \( k \) and hence determine the possible sequences.

**Solution**

\[
\frac{T_2}{T_1} = \frac{T_3}{T_2}
\]

\[
\therefore \frac{k - 1}{k + 1} = \frac{2k - 5}{k - 1}
\]

\[
\therefore (k - 1)^2 = (2k - 5)(k + 1)
\]

\[
\therefore k^2 - 2k + 1 = 2k^2 - 3k - 5
\]

\[
\therefore 0 = k^2 - k - 6
\]

\[
\therefore 0 = (k - 3)(k + 2)
\]

\[
\therefore k = 3 \text{ or } k = -2
\]

For \( k = 3 \):

\[ T_1 = k + 1 = 3 + 1 = 4 \]

\[ T_2 = k - 1 = 3 - 1 = 2 \]

\[ T_3 = 2k - 5 = 2(3) - 5 = 1 \]

The sequence is therefore: \( 4; 2; 1; \ldots \ldots \)

For \( k = -2 \):

\[ T_1 = k + 1 = -2 + 1 = -1 \]

\[ T_2 = k - 1 = -2 - 1 = -3 \]

\[ T_3 = 2k - 5 = 2(-2) - 5 = -9 \]

The sequence is therefore: \( -1; -3; -9; \ldots \ldots \)

**EXERCISE 2**

1. Determine the general term for each of the following geometric sequences:

   (a) \( 2; -1; \frac{1}{2}; \ldots \ldots \)  
   (b) \( 2; 8; 32; \ldots \ldots \)  
   (c) \( -\frac{2}{3}; 2; -6; \ldots \ldots \)  
   (d) \( 1; 0.2; 0.04; \ldots \ldots \)  

2. Determine the 9th term for each of the following geometric sequences:

   (a) \( 128; 64; 32; \ldots \ldots \)  
   (b) \( 0.25; 0.5; 1; \ldots \ldots \)  
   (c) \( \frac{4}{9}; \frac{1}{3}; 4; \ldots \ldots \)  
   (d) \( \frac{2}{3}; -1; \frac{3}{2}; \ldots \ldots \)  
   (e) \( T_k = 3 \left( \frac{2}{3} \right)^{6-k} \)  
   (f) \( T_k = 2400 \left( \frac{1}{2} \right)^{k-1} \)

3. (a) Which term of the geometric sequence \( 2; 6; 18; \ldots \ldots \) is equal to 4374?
   (b) Determine the number of terms in the sequence \( -\frac{3}{8}; \frac{3}{4}; -\frac{3}{2}; \ldots \ldots ; 192 \)

4. Consider the following sequence: \( 0.625; 1.25; 2.5; 5; \ldots \ldots \)
   (a) Determine the value of the 10th term.
   (b) Determine which term will equal 80.

5. Consider the following sequences:
   - sequence 1: \( \sqrt{2}; 2\sqrt{2}; 3\sqrt{2}; \ldots \ldots \)  
   - sequence 2: \( \sqrt{2}; 2; 2\sqrt{2}; \ldots \ldots \)

   (a) Show that sequence 1 is arithmetic.
   (b) Show that sequence 2 is geometric.
   (c) Which term of sequence 1 will be equal to \( \sqrt{200} \)?
   (d) Which term of sequence 2 will be equal to 256?
6. \(x - 4; x + 2; 3x + 1; \ldots\) are the first three terms of a geometric sequence. Determine the sequence if \(x\) is positive.

7. \(t + 1; 1 - t; 1 - 5t; \ldots\) are the first three terms of a geometric sequence.
   (a) Determine the numerical value of \(t\) where \(t \neq 0\).
   (b) Determine the sequence.
   (c) Determine the 10th term.
   (d) Which term equals \(\frac{2}{3}\)?

**DETERMINING THE SUM OF THE TERMS OF A SEQUENCE**

If \(T_1; T_2; T_3; T_4; \ldots\) denotes a sequence then the sum \(T_1 + T_2 + T_3 + T_4 + \ldots\) is called a series. A series is formed by adding the terms of a sequence.

**EXAMPLE 9**

(a) Sequence: \(1; 3; 5; 7; 9; \ldots\)  
   Corresponding series: \(1 + 3 + 5 + 7 + 9 + \ldots\)

(b) Sequence: \(5; 2; -1; -4; \ldots\)  
   Corresponding series: \(5 + 2 + (-1) + (-4) + \ldots\)  
   or \(5 + 2 - 1 - 4 + \ldots\)

We will use the symbol \(S_n\) to denote the sum of the first \(n\) terms of a series.

\[ S_1 = T_1 \]
\[ S_2 = T_1 + T_2 \quad \text{The sum of the first 2 terms} \]
\[ S_3 = T_1 + T_2 + T_3 \quad \text{The sum of the first 3 terms} \]
\[ S_4 = T_1 + T_2 + T_3 + T_4 \quad \text{The sum of the first 4 terms} \]
\[ S_n = T_1 + T_2 + T_3 + \ldots + T_n \quad \text{The sum of the first } n \text{ terms} \]

**EXAMPLE 10**

Consider the series: \(1 + 4 + 7 + 10 + \ldots\)  
Determine:  
(a) \(S_3\)  
(b) \(S_7\)

**Solutions**

(a) \(S_3 = 1 + 4 + 7 = 12\)  
(b) \(S_7 = 1 + 4 + 7 + 10 + 13 + 16 + 19 = 70\)

**EXAMPLE 11**

Determine the sum of the first 7 terms of the series with \(T_k = 5k - 3\).

**Solution**

\[ S_7 = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 \]
\[ S_7 = [5(1) - 3] + [5(2) - 3] + [5(3) - 3] + [5(4) - 3] + [5(5) - 3] + [5(6) - 3] + [5(7) - 3] \]
\[ S_7 = 2 + 7 + 12 + 17 + 22 + 27 + 32 = 119 \]
SERIES AND SIGMA NOTATION

The mathematical symbol $\sum$ is the capital letter S in the Greek alphabet. It is used as the symbol for summing a series. Consider the following series:

$$\sum_{k=1}^{n} T_k = T_1 + T_2 + T_3 + \ldots + T_n$$

This is read as follows:
The sum of all the terms $T_k$ (general term) from $k = 1$ to $k = n$ where $n \in \mathbb{N}$.

EXAMPLE 12

Calculate: $\sum_{k=1}^{7} (5k - 3)$

Solution
Translating this expression is as follows:
Start with $k = 1$ and substitute all integers from 1 up to and including 7 into the expression $5k - 3$. Simplify each substitution and then determine the sum of the numbers.

$$\sum_{k=1}^{7} (5k - 3) = 5(1) - 3 + 5(2) - 3 + 5(3) - 3 + 5(4) - 3 + 5(5) - 3 + 5(6) - 3 + 5(7) - 3$$

= $2 + 7 + 12 + 17 + 22 + 27 + 32$

= 119

EXAMPLE 13

Calculate: $\sum_{r=0}^{7} 3.2^r$

Solution

$$\sum_{r=0}^{7} 3.2^r = 3.2^0 + 3.2^1 + 3.2^2 + 3.2^3 + 3.2^4 + 3.2^5 + 3.2^6 + 3.2^7$$

= $3 + 6 + 12 + 24 + 48 + 96 + 192 + 384$

= 765

Take note that the number of terms in the above series is actually 8 and not 7.

Consider some further examples:
(a) \[ \sum_{k=1}^{8} 2k = 2(1) + 2(2) + 2(3) + 2(4) + 2(5) + 2(6) + 2(7) + 2(8) \]
\[ = 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 \]
\[ = 72 \quad \text{Number of terms: 8} \]
\[ (8 - 1 + 1 = 8) \]

(b) \[ \sum_{k=0}^{8} 2k = 2(0) + 2(1) + 2(2) + 2(3) + 2(4) + 2(5) + 2(6) + 2(7) + 2(8) \]
\[ = 0 + 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 \]
\[ = 72 \quad \text{Number of terms: 9} \]
\[ (8 - 0 + 1 = 9) \]

(c) \[ \sum_{k=2}^{8} 2k = 2(2) + 2(3) + 2(4) + 2(5) + 2(6) + 2(7) + 2(8) \]
\[ = 4 + 6 + 8 + 10 + 12 + 14 + 16 \]
\[ = 70 \quad \text{Number of terms: 7} \]
\[ (8 - 2 + 1 = 7) \]

Notice that the last value you substitute is not necessarily the number of terms that will be added.

Therefore in general for \( \sum_{k=m}^{n} T_k \):

\[ \text{Number of terms} = \text{Top} - \text{Bottom} + 1 = n - m + 1 \]

**EXAMPLE 14**

Determine the number of terms in the following series:

(a) \[ \sum_{n=5}^{b} n^2 \quad (b) \sum_{k=n+1}^{2n} k \]

**Solutions**

(a) number of terms = \( b - 5 + 1 = b - 4 \) terms.
(b) number of terms = \( 2n - (n + 1) + 1 = 2n - n - 1 + 1 = n \) terms.

**EXAMPLE 15**

Expand and calculate \( \sum_{m=3}^{12} 6 \) (The expression next to the sigma is a constant)

**Solution**

\[ \sum_{m=3}^{12} 6 = \sum_{m=3}^{12} 6m^0 = 6(3)^0 + 6(4)^0 + 6(5)^0 + ... + 6(12)^0 \]
\[ = 6 + 6 + 6 + ... + 6 \quad (\text{Number of terms:} 12 - 3 + 1 = 10) \]
\[ = 6 \times 10 \]
\[ = 60 \]
EXAMPLE 16

(a) Write the following series in sigma notation: \(2 + 5 + 8 + \ldots + 83\)

**Solution**

The above series is arithmetic with \(a = 2\) and \(d = 3\)

\[T_n = a + (n - 1)d\]

\[\therefore T_n = 2 + (n - 1)(3)\]

\[\therefore T_n = 2 + 3n - 3\]

\[\therefore T_n = 3n - 1 \quad \text{(general term)}\]

The number of terms in the series can be calculated as follows:

83 is the last term in the sequence and therefore we know that \(T_n = 83\)

\[\therefore 83 = 3n - 1\]

\[\therefore 3n = 84\]

\[\therefore n = 28\]

\[\therefore 2 + 5 + 8 + \ldots + 83 = \sum_{n=1}^{28} (3n - 1)\]

(b) Write the series \(2 + 6 + 12 + 20 + \ldots\) to \(n\) terms in sigma notation.

**Solution**

\[a + b + c \rightarrow 2 \quad 6 \quad 12 \quad 20\]

\[3a + b \rightarrow 4 \quad 6 \quad 8\]

\[2a \rightarrow 2 \quad 2\]

\[2a = 2 \quad 3(1) + b = 4 \quad (1) + (1) + c = 2\]

\[\therefore a = 1 \quad \therefore b = 1 \quad \therefore c = 0\]

\[\therefore T_k = k^2 + k \quad \text{(Use} \ k \text{since} \ n \text{refers to the total number of terms)}\]

\[\therefore 2 + 6 + 12 + 20 + \ldots \text{to} \ n \text{terms} = \sum_{k=1}^{n} (k^2 + k)\]

**EXERCISE 3**

1. Expand and then calculate each of the following:

(a) \[\sum_{r=1}^{5} (3r - 5)\]

(b) \[\sum_{r=3}^{8} \left(\frac{r(r + 1)}{2}\right)\]

(c) \[\sum_{k=1}^{6} k\]

(d) \[\sum_{i=0}^{5} 2^{i-1}\]

(e) \[\sum_{k=3}^{10} 4\]

(f) \[\sum_{p=2}^{7} p^2\]

13
2. Write each of the following series in sigma notation:
   (a)  \( 2 + 4 + 6 + 8 + 10 + 12 \)
   (b)  \( 1 + 8 + 27 + 64 + 125 \)
   (c)  \( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{128} \)
   (d)  \( 3 - 6 + 12 - 24 + \ldots \) to \( n \) terms
   (e)  \( \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \ldots + \frac{24}{25} \)
   (f)  \( 5 + 5 + 5 + 5 + \ldots \) to \( n \) terms
   (g)  \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \) (The Harmonic Series)

SUMMING THE TERMS OF AN ARITHMETIC SERIES

An arithmetic series is the sum of an arithmetic sequence. 5; 7; 9; 11; ... is an arithmetic sequence and 5 + 7 + 9 + 11 + ... is the sum of that sequence.

We will use the formulae below to calculate the sum of a finite arithmetic series:

\[
S_n = \frac{n}{2} [2a + (n-1)d] \quad \text{or} \quad S_n = \frac{n}{2} [a + l] \quad (l \text{ is the last term of the series)}
\]

\[ l = T_n = a + (n-1)d \]

Where do these formulae come from? Consider the following example.

EXAMPLE 17

Evaluate the arithmetic series: \( S_8 = 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 \) (8 terms)

Solution

\[
S_8 = 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 \quad \ldots \text{A}
\]

\[
S_8 = 17 + 15 + 13 + 11 + 9 + 7 + 5 + 3 \quad \ldots \text{B (The series written in reverse)}
\]

\[
2S_8 = 20 + 20 + 20 + 20 + 20 + 20 + 20 \quad \ldots \text{A + B (Add the two expressions)}
\]

\[
2S_8 = 20 \times 8 \quad \text{(number of terms is 8)}
\]

\[
S_8 = \frac{160}{2} = 80
\]

The above method can be used to derive the formula for the sum of the first \( n \) terms of an arithmetic sequence.

THEOREM 1

The sum of the first \( n \) terms of an arithmetic sequence is given by the formula:

\[
S_n = \frac{n}{2} [2a + (n-1)d]
\]

Proof

Let the first term of an arithmetic series be denoted by \( a \) and let \( d \) denote the constant difference.
It is also possible to derive a formula to calculate the sum of a finite arithmetic series consisting of \( n \) terms.

The formula \( S_n = \frac{n}{2} [2a + (n-1)d] \) can be expressed as follows:

\[
S_n = \frac{n}{2} [a + a + (n-1)d]
\]

The last term \( (l) \) is the \( n \)th term and therefore \( l = a + (n-1)d \)

\[
S_n = \frac{n}{2} [a + l]
\]

**EXAMPLE 18**

Find the sum of the first 20 terms of the series \(-1 + 3 + 7 + \ldots\).

**Solution**

The given series is arithmetic with \( a = -1 \) and \( d = 4 \). \( S_{20} = ? \)

\[
S_n = \frac{n}{2} [2a + (n-1)d] \quad \text{(state the formula)}
\]

\[
\therefore S_{20} = \frac{20}{2} [2(-1) + (20-1)(4)] \quad \text{(substitute } n = 20, a = -1 \text{ and } d = 4)\]

\[
\therefore S_{20} = 10(-2 + 76) = 740
\]

**EXAMPLE 19**  (Finding the sum when the number of terms is not known)

Calculate the sum of the following finite series: \(3 + 5 + 7 + \ldots + 111\)

**Solution**

It is necessary to first determine the number of terms in the series before being able to find the sum of the series.

\( a = 3, \ d = 2 \) and \( T_n = 111 \)

\[
T_n = a + (n-1)d \quad \text{(the value of a term has been given)}
\]

\[
\therefore 111 = 3 + (n-1)(2) \quad \text{(substitute } T_n = 111, a = 3 \text{ and } d = 2)\]
\[111 = 3 + 2n - 2\]
\[110 = 2n\]
\[n = 55\]

There are 55 terms in the series.

To calculate the sum of the series use either one of the two formulae:

\[S_n = \frac{n}{2}[a + l]\]  \((\text{last term is known})\) \quad \text{or} \quad \[S_n = \frac{n}{2}[2a + (n-1)d]\]

\[\therefore S_{55} = \frac{55}{2}[3 + 111]\]
\[\therefore S_{55} = 3135\]

\[\therefore S_{55} = \frac{55}{2}[2(3) + (55 - 1)(2)]\]

\[\therefore S_{55} = \frac{55}{2}[6 + 108]\]
\[\therefore S_{55} = 3135\]

**EXAMPLE 20**

Calculate \[\sum_{m=2}^{100} (7 - 2m)\]

**Solution**

Expand so as to identify the type of series:

\[\sum_{m=2}^{100} (7 - 2m) = [7 - 2(2)] + [7 - 2(3)] + [7 - 2(4)] + [7 - 2(5)] + \ldots + [7 - 2(100)]\]

\[= 3 + 1 + (-1) + (-3) + \ldots + (-193)\]

This is an arithmetic series with \(a = 3\) and \(d = T_2 - T_1 = T_3 - T_2 = -2\) and \(l = -193\)

Now determine the number of terms:

\[n = 100 - 2 + 1 = 99\]  \((\text{Number of terms} = \text{Top} - \text{Bottom} + 1)\)

\[\therefore \sum_{m=2}^{100} (7 - 2m) \Rightarrow S_{99} = ?\]  \((\text{an important observation})\)

\[S_n = \frac{n}{2}(a + l)\] \quad \text{or} \quad \[S_n = \frac{n}{2}[2a + (n-1)d]\]

Substitute \(n = 99, a = 3\) and \(l = -193\) \quad \text{Substitute} \quad n = 99, a = 3 \text{ and } d = -2

\[\therefore S_{99} = \frac{99}{2}(3 + (-193))\]
\[\therefore S_{99} = \frac{99}{2}[2(3) + (99 - 1)(-2)]\]
\[\therefore S_{99} = -9405\] \quad \[\therefore S_{99} = -9405\]
EXAMPLE 21  
(Finding $n$ when the sum is given)

How many terms of the arithmetic sequence 4 ; 7 ; 10 ; 13 ; .... will add up to 175?

**Solution**

It is important to emphasize that 175 is not the value of a term but the sum of a certain number of terms ($S_n$).

$\therefore S_n = 175$

$a = 4$, $d = 3$, and $S_n = 175$

$S_n = \frac{n}{2} [2a + (n-1)d]$

$\therefore 175 = \frac{n}{2} [2(4) + (n-1)(3)]$  
(substitute $S_n = 175$, $a = 4$ and $d = 3$)

$\therefore 175 = \frac{n}{2} [8 + 3n - 3]$

$\therefore 175 = \frac{n}{2} [3n + 5]$

$\therefore 175 \times 2 = 3n^2 + 5n$

$\therefore 350 = 3n^2 + 5n$

$\therefore 0 = 3n^2 + 5n - 350$

$\therefore 0 = (n-10)(3n+35)$

$\therefore n = 10 \text{ or } 3n = -35$

$\therefore n = 10 \text{ or } n = -\frac{35}{3}$

But $n \neq -\frac{35}{3}$

$\therefore n = 10$

$\therefore S_{10} = 175$

EXAMPLE 22

Determine $n$ if $\sum_{r=1}^{n} (6r - 1) = 456$

**Solution**

The sum of the series is given and you are required to find the number of terms that should be added give a sum of 456.

$\sum_{r=1}^{n} (6r - 1) = [6(1) - 1] + [6(2) - 1] + [6(3) - 1] + ... + [6(n) - 1] = 456$

$= 5 + 11 + 17 + ... + (6n - 1) = 456$

This is an arithmetic series with $a = 5$ and $d = T_2 - T_1 = T_3 - T_2 = 6$

The number of terms is $n - 1 + 1 = n$

$\therefore \sum_{r=1}^{n} (6r - 1) = 456 \quad \Rightarrow \quad S_n = 456$
\[ S_n = \frac{n}{2} (2a + (n-1)d) \]
\[ \therefore 456 = \frac{n}{2} (2(5) + (n-1)(6)) \] (substitute \( S_n = 456, a = 3 \) and \( d = 6 \))
\[ \therefore 456 = \frac{n}{2} (10 + 6n - 6) \]
\[ \therefore 456 = \frac{n}{2} (4 + 6n) \] \[ \therefore 456 = 2n + 3n^2 \]
\[ \therefore 0 = 3n^2 + 2n - 456 \]
\[ \therefore (3n + 38)(n - 12) = 0 \] (Note: The quadratic formula may be used)
\[ \therefore 3n = -38 \text{ or } n = 12 \]
\[ \therefore n = -\frac{38}{3} \]
But \( n \neq -\frac{38}{3} \)
\[ \therefore n = 12 \]

**EXERCISE 4**

1. Find the sum of each arithmetic series:
   (a) \[ 5 + 8 + 11 + \ldots \text{ to 20 terms} \]
   (b) \[ 10 + 7 + 4 + \ldots \text{ to 32 terms} \]
   (c) \[ -14 - 9 - 4 + \ldots \text{ to 100 terms} \]

2. (a) Calculate \( \sum_{k=1}^{100} (3k + 2) \)
    (b) Calculate \( \sum_{m=2}^{50} (5 - 2m) \)
    (c) Calculate \( \sum_{i=0}^{40} (4i - 1) \)

3. (a) How many terms are there in the series \[ -3 + 1 + 5 + \ldots + 313? \]
    (b) What is the sum of the series?

4. (a) How many terms are there in the series \[ \frac{1}{2} + 2 + 3 \cdot \frac{1}{2} + \ldots + 32? \]
    (b) What is the sum of the series?

5. Find the sum of the following arithmetic series:
   (a) \[ 6 - 5 - 16 - \ldots - 115 \]
   (b) \[ 21 + 23 + 25 + \ldots + 255 \]
   (c) \[ 13 + 6 - 1 - \ldots - 106 \]

6. Find the sum of the first 500 odd numbers.

7. (a) How many terms of the arithmetic sequence 3 ; 11 ; 19 ; \ldots \text{will add up to 1580?} \)
    (b) How many terms of the arithmetic sequence \(-3 ; 2 ; 7 ; \ldots \text{will add up to 1089?} \)
    (c) Determine \( n \) if: \[ 4 + 13 + 22 + \ldots (\text{to } n \text{ terms}) = 539 \]

8. Determine \( m \) in each of the following arithmetic series:
   (a) \[ \sum_{k=1}^{m} (7k + 5) = 1287 \]
   (b) \[ \sum_{p=1}^{m} (2p - 7) = 1015 \]
   (c) \[ \sum_{i=0}^{m} (1 - 3i) = -671 \]
9. What is the greatest number of terms for which the series \( \sum_{k=1}^{n} (k+1) \) will have a value less than 65?

10. Calculate \( \frac{3 + 6 + 9 + \ldots + 402}{2 + 4 + 6 + \ldots + 402} \)

11. (a) Calculate \( \sum_{i=1}^{10} 1 \) 
(b) Calculate \( \sum_{i=1}^{50} 1 \) 
(c) Hence show that \( \sum_{i=1}^{n} 1 = n \) (An important result to remember)

12. An athlete trains by running 600 metres on the first day, 900 metres on the second, 1200 metres on the third and so forth. 
(a) How far does he run on the 15th day? 
(b) What is the total distance that he will run in 15 days? 
(c) How long will it be before he can run a marathon of 42km?

13. Shaun can do 27 push-ups per minute. Each week he improves his performance by 3 push-ups per minute. Mpho can do 20 push-ups per minute and he increases his performance by 4 push-ups per minute each week. 
(a) How many push-ups will they each do per minute in the fitness competition which is in 10 weeks’ time? 
(b) After how many weeks will their number of push-ups per minute be the same? 
(c) While training for the competition, Shaun spent 45 minutes doing push-ups per week. How many push-ups does he do altogether in the 10 weeks?

14. A ladder has 50 rungs. The bottom rung is 1 m long. Each rung is 12.5 mm shorter than the rung beneath it. Determine the total length of wood required to make 50 rungs.

**SUMMING THE TERMS OF A GEOMETRIC SERIES**

A geometric series is the sum of a geometric sequence. \( 3 ; 6 ; 12 ; 24 ; \ldots \) is a geometric sequence and the sum of that sequence, \( 3 + 6 + 12 + 24 + \ldots \) is a geometric series.

We will use the formulae below to calculate the sum of a finite geometric series:

\[
S_n = \frac{a(r^n - 1)}{r - 1}, \quad r \neq 1 \\
S_n = \frac{a(1 - r^n)}{1 - r}, \quad r \neq 1
\]

Where do these formulae come from? Consider the following example:
EXAMPLE 23

Evaluate the geometric series: \( S_6 = 4 + 12 + 36 + 108 + 324 + 972 \) (6 terms)

Solution

\( S_6 = 4 + 12 + 36 + 108 + 324 + 972 \) \( \ldots \text{A} \) \((r = 3)\)

\( \therefore 3S_6 = 12 + 36 + 108 + 324 + 972 + 2916 \) \( \ldots \text{B} \) (The series multiplied by \( r \))

\( \therefore S_6 - 3S_6 = 4 + 0 + 0 + 0 + 0 - 2916 \) \( \ldots (A - B) \)

\( \therefore -2S_6 = -2912 \)

\( \therefore S_6 = \frac{-2912}{-2} = 1456 \)

The above method can be used to derive the formula for the sum of the first \( n \) terms of a geometric sequence.

THEOREM 2

The sum of the first \( n \) terms of a geometric sequence is given by the formula

\[ S_n = a \frac{r^n - 1}{r - 1} \] \( \text{where } r \neq 1 \)

\[ S_n = a \frac{1 - r^n}{1 - r} \] \( \text{where } r \neq 1 \)

Proof

Let the first term of a finite geometric series be denoted by \( a \) and let \( r \) denote the constant ratio.

\( S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \) \( \ldots \text{A} \)

\( \therefore r \times S_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n \) \( \ldots \text{B} \) (The series multiplied by \( r \))

Subtract B from A

\( S_n - rS_n = a + 0 + 0 + \ldots + 0 - ar^n \) \( \ldots (A - B) \)

\( \therefore S_n(1-r) = a - ar^n \)

\( \therefore S_n(1-r) = a(1-r^n) \)

\( \therefore S_n = a \frac{1-r^n}{1-r} \) \( \text{where } r \neq 1 \)

Similarly, by subtracting \( S_n \) from \( rS_n \), the following formula can be derived:

\[ S_n = a \frac{r^n - 1}{r - 1} \] \( \text{where } r \neq 1 \)

EXAMPLE 24

Find the sum of the first 12 terms of the series \( \frac{2}{3} + 2 + 6 + \ldots \)

Solution

This is a geometric series with \( a = \frac{2}{3} \) and \( r = 3 \). \( S_{12} = ? \)
\[ S_n = \frac{a(r^n - 1)}{r - 1} \]  
(state the formula)

\[ \therefore S_{12} = \frac{2}{3} \left(\frac{(3)^{12} - 1}{(3) - 1}\right) \]  
(substitute \( n = 12, a = \frac{2}{3} \) and \( r = 3 \))

\[ \therefore S_{12} = 177146.6 \]

**EXAMPLE 25**  (Finding the sum when the number of terms is not known at first)

Calculate the sum of the following finite series \( 0.25 + 0.5 + 1 + 2 + \ldots + 256 \).

**Solution**

It is necessary to first calculate the number of terms in the series before being able to determine the sum of the series.

\( a = 0.25, \ r = 2 \) and \( T_n = 256 \)

\[ T_n = ar^{n-1} \]  
(state the formula)

\[ \therefore 256 = (0.25)(2)^{n-1} \]  
(substitute \( T_n = 256, a = 0.25 \) and \( r = 2 \))

\[ \therefore 1024 = 2^{n-1} \]

\[ \therefore 2^{10} = 2^{n-1} \]

\[ \therefore 10 = n - 1 \]

\[ \therefore n = 11 \]

There are 11 terms in the series.

In order to calculate the sum of the series, either one of the two formulae may be used:

\[ S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1-r^n)}{1-r} \]

\[ \therefore S_{11} = \frac{0.25((2)^{11} - 1)}{(2) - 1} \]

\[ \therefore S_{11} = 511.75 \]

\[ \therefore S_{11} = 511.75 \]

**EXAMPLE 26**

Calculate \( \sum_{i=0}^{10} 3 \cdot (-2)^i \)

**Solution**

\[ \sum_{i=0}^{10} 3 \cdot (-2)^i = \left[ 3 \cdot (-2)^{0-1} \right] + \left[ 3 \cdot (-2)^{1-1} \right] + \left[ 3 \cdot (-2)^{2-1} \right] + \ldots + \left[ 3 \cdot (-2)^{10-1} \right] \]

\[ = -\frac{3}{2} + 3 + (-6) + \ldots + 786\ 432 \]
This is a geometric series with \( a = \frac{-3}{2} \) and \( r = -2 \)

The number of the terms is:

\[ n = 19 - 0 + 1 = 20 \quad \text{(Number of terms = Top - Bottom + 1)} \]

\[ \sum_{i=0}^{19} 3(-2)^{i-1} \Rightarrow S_{20} = ? \]

\[ S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{(state the formula and substitute} \ a = \frac{-3}{2}, \ r = -2 \text{ and} \ n = 20) \]

\[ \therefore S_{20} = \frac{\left(\frac{-3}{2}\right)((-2)^{20} - 1)}{(-2) - 1} = 524,287.5 \]

**EXAMPLE 27** \ (Finding \( n \) when the sum is given)

How many terms of the geometric sequence \(-1; 2; -4; 8; \ldots\) will add up to \(349,525\)?

**Solution**

It is important to emphasize that \(349,525\) is not the value of a term but the sum of a certain number of terms \( (S_n) \).

\[ a = -1, \ r = -2 \text{ and } S_n = 349,525 \]

\[ S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{(state the formula)} \]

\[ \therefore 349,525 = \frac{(-1)((-2)^n - 1)}{(-2) - 1} \quad \text{(substitute } S_n = 349,525, \ a = -1 \text{ and} \ r = -2) \]

\[ \therefore 349,525 = \frac{(-1)((-2)^n - 1)}{-3} \]

\[ \therefore 349,525 \times (-3) = (-1)((-2)^n - 1) \]

\[ \therefore 1,048,575 = (-2)^n - 1 \]

\[ \therefore 1,048,576 = (-2)^n \]

\[ \therefore (-2)^{20} = (-2)^n \]

\[ \therefore n = 20 \]

\[ \therefore \text{The first 20 terms must be added to give 349,525} \]
EXAMPLE 28

Determine \( m \):

\[
\sum_{k=2}^{m} \frac{1}{15} (3)^{k-1} = 24 \frac{1}{5}
\]

**Solution**

The sum of the series is given. You are required to find the number of terms that should be added to give a sum of \( 24 \frac{1}{5} \).

\[
\sum_{k=2}^{m} \frac{1}{15} (3)^{k-1} = \left[ \frac{1}{15} (3)^{2-1} \right] + \left[ \frac{1}{15} (3)^{3-1} \right] + \left[ \frac{1}{15} (3)^{4-1} \right] + \ldots + \left[ \frac{1}{15} (3)^{m-1} \right] = 24 \frac{1}{5}
\]

\[
= 0,2 + 0,6 + 1,8 + \ldots + \left( \frac{1}{15} (3)^{m-1} \right) = 24 \frac{1}{5}
\]

This is a geometric series with \( a = 0,2 \) and \( r = 3 \)

The number of the terms is: \( m - 2 + 1 = m - 1 \)

\[
\therefore \sum_{k=2}^{m} \frac{1}{15} (3)^{k} = 24 \frac{1}{5} \quad \Rightarrow \quad S_{m-1} = 24 \frac{1}{5}
\]

\[
S_{m-1} = \frac{a(r^{m-1} - 1)}{r - 1}
\]

\[
\therefore 24 \frac{1}{5} = \frac{0,2\left((3)^{m-1} - 1\right)}{(3) - 1}
\]

(substitute \( S_n = 24 \frac{1}{5} \), \( a = 0,2 \) and \( r = 3 \))

\[
\therefore 24,2 = \frac{0,2\left((3)^{m-1} - 1\right)}{2}
\]

\[
\therefore 24,2 \times 2 = 0,2\left((3)^{m-1} - 1\right)
\]

\[
\therefore \frac{24,2 \times 2}{0,2} = (3)^{m-1} - 1
\]

\[
\therefore 242 = (3)^{m-1} - 1
\]

\[
\therefore 243 = 3^{m-1}
\]

\[
\therefore 3^3 = 3^{m-1}
\]

\[
\therefore 5 = m - 1
\]

\[
\therefore 6 = m
\]

**EXERCISE 5**

1. Determine the sum of each geometric series (use an appropriate formula):
   
   (a) \[ \frac{2}{27} + \frac{2}{9} + \frac{2}{3} + \ldots \text{ to 9 terms} \]
   
   (b) \[ -64 + 32 - 16 + \ldots \text{ to 10 terms} \]
2. Calculate the sum of the first 12 terms of the geometric sequence
   \[ \frac{1}{4}, -\frac{1}{2}, 1; \ldots \]

3. (a) How many terms are there in the series \(-64 - 32 - 16 - \ldots - \frac{1}{32}\)?
   (b) What is the sum of the series?

4. Evaluate: \(-9 - 6 - 4 - \ldots - 1\frac{5}{27}\)

5. (a) Calculate \(\sum_{k=0}^{10} \frac{1}{50} (5)^{k-1}\)
   (b) Calculate \(\sum_{m=3}^{11} \left(\frac{1}{2}\right)^{m-4}\)
   (c) Calculate \(\sum_{i=0}^{10} 3^{4-i}\)

6. (a) How many terms of the geometric sequence 64; 32; 16; ..... will add up to \(127\frac{1}{2}\)?
   (b) How many terms of the geometric sequence \(\frac{1}{27}; \frac{1}{9}; \frac{1}{3}; \ldots\) yield a sum of \(\frac{364}{27}\)?
   (c) Determine \(n\) if: \(3 + 6 + 12 + \ldots\) (to \(n\) terms) = 765

7. Determine \(m\) in each of the following arithmetic series:
   (a) \(\sum_{k=1}^{m} (2)^{8-k} = 255\frac{1}{2}\)
   (b) \(\sum_{p=1}^{m} (-8)\left(\frac{1}{2}\right)^{p-1} = -15\frac{3}{4}\)
   (c) \(\sum_{i=1}^{m} 5(1)^{i-2} = 500\)

8. What is the least value of \(p\) for which the series \(\sum_{k=1}^{p} \frac{1}{16} (2)^{k-2} > 31\)?

9. Mr Deeds gives his son R1 on his 1st birthday, R2 on his 2nd birthday, R4 on his 3rd birthday, R8 on his 4th birthday and so forth.
   (a) How much will he give his son on his 18th birthday?
   (b) Find the total amount of money he will have given him by that stage?

**PROBLEMS INVOLVING SIMULTANEOUS EQUATIONS**

**EXAMPLE 29**

(a) Determine the first 3 terms of an arithmetic sequence if the 5th term of the sequence is 12 and the 14th term is \(-33\).
   (b) Hence determine the 40th term.

**Solution**

\(\_ ; \_ ; \_ ; \_ ; 12 ; \_ ; \_ ; \_ ; \_ ; \_ ; \_ ; \_ ; \_ ; \_ ; \_ ; \_ ; \_ ; \_ ; \_ ; \_ ; \_ ; -33 ; \ldots \)

To find the first three terms of the sequence you have to find the value of \(a\) and the constant difference \(d\) so as to generate the sequence.
\( T_3 = -33 \) and \( T_5 = 12 \)

\[ \therefore T_{14} = a + (14 - 1)d \quad \text{and} \quad T_5 = a + (5 - 1)d \]

\[ \therefore a + (14 - 1)d = -33 \quad \text{and} \quad \therefore a + (5 - 1)d = 12 \]

\[ \therefore a + 13d = -33 \quad \ldots \text{A} \quad \text{and} \quad \therefore a + 4d = 12 \quad \ldots \text{B} \]

Solve the two equations simultaneously:

\[ a + 13d = -33 \quad \ldots \text{A} \]
\[ a + 4d = 12 \quad \ldots \text{B} \]

\[ \therefore 9d = -45 \quad \text{A} - \text{B} \]

\[ \therefore d = -5 \]

\[ \therefore a + 13(-5) = -33 \quad \text{Substitute} \ d = -5 \ \text{into A} \]
\[ \therefore a - 65 = -33 \]
\[ \therefore a = 32 \]

\[ \therefore T_1 = 32, \ T_2 = 32 + (-5) = 27 \ \text{and} \ T_3 = 27 + (-5) = 22 \]

\[ \therefore \text{The first three terms of the sequence are} \ 32; 27; 22 \]

\[ \therefore \text{T}_{30} = 32 + (40 - 1)(-5) = -163 \]

**EXAMPLE 30**

The sum of the first 12 terms of an arithmetic series is 96. The 3rd and 6th terms add up to 12. Determine the first term and the common difference.

**Solution**

\[ S_{12} = 96 \quad \text{and} \quad T_3 + T_6 = 12 \]

\[ S_n = \frac{n}{2} \left( 2a + (n - 1)d \right) \quad \text{T}_n = a + (n - 1)d \]

\[ \therefore \frac{12}{2} \left( 2a + (12 - 1)d \right) = 96 \quad \therefore \left( a + (3 - 1)d \right) + \left( a + (6 - 1)d \right) = 12 \]

\[ \therefore 6 \left( 2a + 11d \right) = 96 \quad \left( a + 2d \right) + \left( a + 5d \right) = 12 \]

\[ \therefore 2a + 11d = 16 \quad \ldots \text{A} \quad \text{and} \quad 2a + 7d = 12 \quad \ldots \text{B} \]

Solve simultaneously:

\[ 2a + 11d = 16 \quad \ldots \text{A} \]
\[ 2a + 7d = 12 \quad \ldots \text{B} \]

\[ \therefore 4d = 4 \quad \text{A} - \text{B} \]

\[ \therefore d = 1 \]

Substitute \( d = 1 \) into A

\[ \therefore 2a + 11(1) = 16 \]

\[ \therefore 2a + 11 = 16 \]

\[ \therefore 2a = 5 \]

\[ \therefore a = \frac{5}{2} \]

\[ \therefore \text{The first term is} \ \frac{5}{2} \ \text{and the constant difference is} \ 1 \]
EXAMPLE 31

Determine the first three terms of the geometric sequence of which the 7th term is 1458 and the 4th term is 54.

Solution

To find the first three terms you have to find the value of $a$ and the constant ratio $r$ to generate the sequence.

$T_4 = 54$ and $T_7 = 1458$

$T_n = ar^{n-1}$ (general term of a geometric sequence)

$T_4 = ar^{4-1}$ and $T_7 = ar^{7-1}$

∴ $ar^3 = 54$ and $ar^6 = 1458$

∴ $ar^3 = 54 \ldots \text{ A}$ and $ar^6 = 1458 \ldots \text{ B}$

Solving simultaneously:

∴ $ar^6 = 1458 \ldots \text{ B}$

∴ $ar^3 = 54 \ldots \text{ A}$

∴ $r^3 = 27 \ldots \text{ B + A}$

∴ $r = 3$

∴ $a(3)^3 = 54 \quad \text{(substitute } r = 3 \text{ into A)}$

∴ $27a = 54$

∴ $a = 2$

∴ The first three terms are: 2; 2×3; 2×3² = 2; 6; 18

EXAMPLE 32

The constant ratio of a geometric series is $-2 \frac{1}{2}$. The sum of the first four terms is $17 \frac{2}{5}$. Calculate the first term.

Solution

$r = -2 \frac{1}{2} = -\frac{5}{2}$ and $S_4 = 17 \frac{2}{5} = \frac{87}{5}$

$S_n = \frac{a(r^n - 1)}{r - 1}$

∴ $S_4 = \frac{a(r^4 - 1)}{r - 1}$ (substitute $n = 4$)

∴ $\frac{87}{5} = \frac{a\left((-\frac{5}{2})^4 - 1\right)}{(-\frac{5}{2})-1}$ (substitute $S_4 = \frac{87}{5}$ and $r = -\frac{5}{2}$)
\[ \therefore 17,4 = a(-10,875) \]
\[ \frac{-17,4}{-10,875} = a \]
\[ \therefore a = -1,6 \]

**EXERCISE 6**

1. Determine the first three terms of each of the following arithmetic sequences of which:
   (a) the 3rd term of the sequence is 23 and the 26th term is 230.
   (b) the 5th term of the sequence is 19 and the 15th term is 59.
2. The 15th and 3rd terms of an arithmetic sequence are 100 and 28 respectively. Determine the 100th term.
3. The 13th and 7th terms of an arithmetic sequence are 15 and 51 respectively.
   (a) Which term is equal to –21?
   (b) Show that 66 is not a term of the sequence.
4. (a) If \( T_4 = -4 \) and \( S_{16} = 24 \) of an arithmetic series, determine the first term and the constant difference of the series.
   (b) The fifth term of an arithmetic sequence is 0 and thirteenth term is 12. Determine the sum of the first 21 terms of sequence.
   (c) The 1st term of an arithmetic sequence is 6 and the sum of the first five terms is 250. Calculate the 12th term of the sequence.
5. The first term and the last term of an arithmetic series is 5 and 61 respectively while the sum of all the terms is 957. Determine the number of terms in the series.
6. The sum of the first 10 terms of an arithmetic series is 145 and the sum of its fourth and ninth term is five times the third term. Determine the first term and constant difference.
7. Given is the series \( 1 + 2 + 3 + 4 + 5 + \ldots + n \)
   (a) Show that \( S = \frac{n(n + 1)}{2} \).
   (b) Find the sum of the first 1001 terms excluding all multiples of 7.
8. Determine the first three terms of each of the following geometric sequences of which:
   (a) the 6th term is 28 and 11th term is 896.
   (b) the 2nd term is 3 and the 4th term is \( 6\frac{3}{4} \).
9. The 9th term and the 6th term of a geometric sequence are 80 and 10 respectively.
   (a) Find the first term and the constant ratio.
   (b) Find the number of terms if the last term is 5120.
10. If \( T_3 = \frac{15}{16} \), \( T_6 = \frac{5}{18} \) and the last term is \( \frac{40}{729} \), find the number of terms in the sequence if the sequence is geometric.
11. (a) In a geometric sequence, \( T_5 = -243 \) and \( T_3 = 72 \). Determine:
    (1) the constant ratio.
    (2) the sum of the first 10 terms.
(b) The constant ratio of a geometric sequence is \( \frac{1}{2} \). The 8th term of the same sequence is \( \frac{5}{32} \). Determine the sum of the first 8 terms.

12. The sum of the first 4 terms of a geometric series is 15 and the sum of the next 4 terms is 240. Determine the positive constant ratio.

THE SUM TO INFINITY OF A CONVERGENT GEOMETRIC SERIES

Consider the following infinite geometric series:

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots
\]

The terms of this series can be added in the following manner and the value towards which the decimals approach or converge to can then be investigated.

\[
S_1 = \frac{1}{2} = 0,5 \\
S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 0,75 \\
S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} = 0,875 \\
S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16} = 0,9375 \\
S_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32} = 0,96875 \\
S_6 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} = \frac{63}{64} = 0,984375
\]

As we continue to add the terms in this way, it seems that the decimal values are tending towards a value of 1. Let’s explore this in more detail.

\[
S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{2^n} = \frac{\left(\frac{1}{2}\right)\left[1 - \left(\frac{1}{2}\right)^n\right]}{1 - \left(\frac{1}{2}\right)} = 1 - \left(\frac{1}{2}\right)^n
\]

The expression \( 1 - \left(\frac{1}{2}\right)^n \) can be represented graphically as the graph of an exponential function \( f(n) = 1 - \left(\frac{1}{2}\right)^n \). By referring to the graph of this function, it can be seen that as the values of \( n \) tend toward infinity, the function values of \( 1 - \left(\frac{1}{2}\right)^n \) tend towards 1 (see graph below).
Clearly as \( n \to \infty \), the expression \( 1 - \left(\frac{1}{2}\right)^n \to 1 \).

Therefore as \( n \to \infty \), it is clear \( S_n \to 1 \)

The progressive sums of the convergent geometric series clearly approach the number 1, which is referred to as the sum to infinity of the convergent geometric series.

**Note**: In general, \( r^n \to 0 \) as \( n \to \infty \) provided that the constant ratio lies in the interval \(-1 < r < 1\). The series will converge to a number (limit) referred to as the **sum to infinity**.

**Divergent geometric series**

Consider the geometric series \( 2 + 4 + 6 + 8 + 16 + \ldots \)

\begin{align*}
S_1 &= 2 \\
S_2 &= 2 + 4 = 6 \\
S_3 &= 2 + 4 + 8 = 14 \\
S_4 &= 2 + 4 + 8 + 16 = 30 \\
& \vdots \\
S_n &= 2 + 4 + 8 + 16 + \ldots + 2^n = \frac{(2)(2^n - 1)}{(2) - 1} = (2)(2^n - 1) = 2.2^n - 2
\end{align*}

The expression \( 2.2^n - 2 \) can be represented graphically as the graph of an exponential function \( f(n) = 2.2^n - 2 \). By referring to the graph of this function, it can be seen that as the values of \( n \) tend toward infinity, the function values of \( 2.2^n - 2 \) tend towards \( \infty \) (see graph below).

![Graph of exponential function](image)

Clearly as \( n \to \infty \), the \( 2.2^n - 2 \to \infty \).

Therefore as \( n \to \infty \), it is clear \( S_n \to \infty \)

The progressive sums of the geometric series do not approach a specific value and therefore the sum to infinity doesn’t exist. The series is said to **diverge**.

**Note**: In general, \( r^n \to \infty \) as \( n \to \infty \) provided that \( r < -1 \) or \( r > 1 \).
Summary of the cases above

Case 1  In a convergent geometric series where \(-1 < r < 1\), the sum to infinity exists.

Case 2  In a divergent geometric series where \(r < -1\) or \(r > 1\), the sum to infinity doesn’t exist.

**THEOREM 3**

In a convergent geometric series in which \(-1 < r < 1\), the sum to infinity is given by the formula: 

\[ S_\infty = \frac{a}{1-r} \]

**Proof** (Not for examination purposes)

For a geometric series with first term \(a\) and constant ratio \(r\):

\[ S_n = \frac{a(1-r^n)}{1-r} \]

\[ \therefore S_n = \frac{a - ar^n}{1-r} \]

\[ \therefore S_n = \frac{a}{1-r} - \frac{ar^n}{1-r} \]

If \(-1 < r < 1\), \(r^n \to 0\) as \(n \to \infty\)

\[ \therefore ar^n \to 0\ as\ n \to \infty \]

\[ \therefore \frac{ar^n}{1-r} \to 0\ as\ n \to \infty \]

\[ \therefore S_n \to \frac{a}{1-r} - 0\ as\ n \to \infty \]

\[ \therefore S_n \to \frac{a}{1-r}\ as\ n \to \infty \]

\[ \therefore \text{The sum gets closer and closer to} \ \frac{a}{1-r}\ \text{as} \ n \ \text{increases.} \]

\[ \therefore S_n = \frac{a}{1-r} \]

The formula for the sum to infinity of a convergent geometric series with first term \(a\) and constant ratio \(r\) is: 

\[ S_\infty = \frac{a}{1-r}\ where \ -1 < r < 1 \]
EXAMPLE 33
Calculate the sum to infinity of the geometric series \( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \).

Solution
The series is geometric with \( a = \frac{1}{2} \) and \( r = \frac{1}{2} \).
Since \(-1 < \frac{1}{2} < 1\), the series converges and the sum to infinity exists.
\[
S_\infty = \frac{a}{1 - r} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1
\]
(substitute \( a = \frac{1}{2} \) and \( r = \frac{1}{2} \)).

EXAMPLE 34
Calculate: \( \sum_{n=1}^{\infty} 2.10^{1-n} \) (if it exists)

Solution
\[
\sum_{n=1}^{\infty} 2.10^{1-n} = [2.10^{1-1}] + [2.10^{1-2}] + [2.10^{1-3}] + \ldots
= 2 + 0,2 + 0,02 + \ldots
\]
This is an infinite geometric series with \( a = 2 \) and \( r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = 0,1 \).
Since \(-1 < 0,1 < 1\), the series converges.
\[
S_\infty = \frac{a}{1 - r} = \frac{2}{1 - 0,1} = \frac{20}{9} = 2,\overline{2}
\]

EXAMPLE 35
Convert the recurring decimal \( 2,\overline{53} \) into a common fraction.

Solution
First write the recurring decimal as a geometric series:
\( 2,\overline{53} = 2,535353... = 2 + (0,53 + 0,0053 + 0,000053 + \ldots) \)
In the brackets there is an infinite geometric series with \( a = 0,53 \) and \( r = 0,01 \).
Since \(-1 < 0,01 < 1\), the series converges.
\[
S_\infty = \frac{a}{1 - r} = \frac{0,53}{1 - (0,01)} = \frac{53}{99}
\]
(substitute \( a = 0,53 \) and \( r = 0,01 \)).
\[
\therefore 2,\overline{53} = 2 + \frac{53}{99} = 2 \frac{53}{99}
\]
EXAMPLE 36
Consider the infinite geometric series $p + p(p+1) + p(p+1)^2 + \ldots$.

(a) For what values of $p$ will the series converge?
(b) Assuming the series is convergent, calculate the sum to infinity.

Solutions
(a) In this series $a = p$ and $r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = p + 1$

The series will converge if $-1 < r < 1$

$\therefore -1 < p + 1 < 1$
$\therefore -1 - 1 < p < 1 - 1$
$\therefore -2 < p < 0$

(b) $S_n = \frac{a}{1 - r}$

$\therefore S_n = \frac{p}{1 - (p + 1)}$ (substitute $a = p$ and $r = p + 1$)

$\therefore S_n = \frac{p}{1 - p - 1}$

$\therefore S_n = \frac{p}{-p} = -1$

EXAMPLE 37
A convergent geometric series has a second term of 8 and a sum to infinity of 36. Determine the possible constant ratio(s).

Solution

$$T_2 = 8 \quad \text{and} \quad S_\infty = 12$$

$$T_2 = ar^{2-1} = ar \quad \text{and} \quad \frac{a}{1 - r} = 36$$

$\therefore ar = 8$ \quad $a = 36(1 - r)$

$\therefore a = \frac{8}{r}$

$\therefore 8 = 36(1 - r)$

$\therefore 8 = 36r(1 - r)$

$\therefore 8 = 36r - 36r^2$

$\therefore 36r^2 - 36r + 8 = 0$

$\therefore 9r^2 - 9r + 2 = 0$

$\therefore (3r - 2)(3r - 1) = 0$

$\therefore r = \frac{2}{3} \text{ or } r = \frac{1}{3}$
EXAMPLE 38  (Real life application)

A rubber ball falls from a height of one metre above the ground. It bounces back to a height of $\frac{9}{10}$ of a metre. It falls again and returns to a height of $\frac{9}{10}$ of the immediate previous height. If this action were to continue indefinitely (theoretically), show that the total distance covered by the ball could never exceed 19 metres.

Solution

\[ D = 1 + 2 \left( \frac{9}{10} \right) + 2 \left( \frac{9}{10} \right)^2 + 2 \left( \frac{9}{10} \right)^3 + 2 \left( \frac{9}{10} \right)^4 + \ldots \]

\[ D = 1 + 2 \left[ \left( \frac{9}{10} \right) + \left( \frac{9}{10} \right)^2 + \left( \frac{9}{10} \right)^3 + \left( \frac{9}{10} \right)^4 + \ldots \right] \]

\[ D = 1 + 2 \left[ \frac{\frac{9}{10}}{1 - \frac{9}{10}} \right] = 1 + 2 \left[ \frac{9}{10} \right] = 1 + 18 = 19 \text{m} \]

Since the progressive sums of the terms of the series approach 19 metres, it means that it will be impossible for the distance to ever exceed 19 metres.

EXERCISE 7

1. Find the sum of each of the following infinite geometric series:
   (a) $2 + \frac{2}{3} + \frac{2}{9} + \ldots$  
   (b) $-64 + 32 - 16 + \ldots$  
   (c) $24 - 4 + \frac{2}{3} - \ldots$

2. (a) Calculate $\sum_{m=1}^{\infty} 8(2)^{-2m}$  
   (b) Calculate $\sum_{m=0}^{\infty} 3 \left( -\frac{1}{2} \right)^m$  
   (c) Calculate $\sum_{i=0}^{\infty} \frac{1}{10^i}$

3. Convert each of the following recurring decimals to a common fraction by first writing it as a geometric series.
   (a) $0,\overline{23}$  
   (b) $4,\overline{2}$  
   (c) $0,\overline{54}$
4. For which values of \( x \) will the following geometric series converge?
   (a) \( 1 + (2x + 1) + (2x + 1)^2 + \ldots \)
   (b) \( 2 - x + (2 - x)^2 + (2 - x)^3 + \ldots \)
   (c) \( \sum_{i=0}^{\infty} 4(3-x)^i \)

5. (a) The first term of a geometric series is 124. The sum to infinity is 64. Find the common ratio.
   (b) For a geometric series with \( r = 0.22 \) and \( S_n = 20 \). Find the first term.

6. In a geometric sequence, the second term is \( -\frac{2}{3} \) and the sum to infinity of the sequence is \( \frac{3}{5} \). Find the sequence.

7. The sum to infinity of a convergent geometric series is 32 and \( r = \frac{1}{2} \). Find the difference between the sum to infinity and the sum of the first five terms.

8. A specific tree grows 1.5 m in the 1st year. Its growth each year thereafter is \( \frac{2}{3} \) of its growth in the previous year. What is the greatest height it can reach?

9. A ball bounces \( \frac{1}{2} \) of its previous height on each bounce. Find the total distance covered by the ball, until it comes to rest, if it is dropped from a height of 20 metres.

**Summary of all formulae**

<table>
<thead>
<tr>
<th>Arithmetic</th>
<th>Geometric</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d = T_{k+1} - T_k ) ( (d = T_2 - T_1 = T_3 - T_2 = \ldots) )</td>
<td>( r = \frac{T_{k+1}}{T_k} ) ( (r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = \ldots) )</td>
</tr>
<tr>
<td>( T_n = a + (n-1)d )</td>
<td>( T_n = ar^{n-1} )</td>
</tr>
<tr>
<td>( S_n = \frac{n}{2}[2a + (n-1)d] ) ( ) or ( S_n = \frac{n}{2}(a + l) )</td>
<td>( S_n = \frac{a(r^n - 1)}{r - 1} ) ( ) or ( S_n = \frac{a(1-r^n)}{1-r} ) ( ) where ( r \neq 1 )</td>
</tr>
<tr>
<td>( S_n = \frac{a}{1-r} ) ( (-1 &lt; r &lt; 1) ) and ( r \neq 0 )</td>
<td></td>
</tr>
</tbody>
</table>
REVISION EXERCISE

1. Write each of the following series in sigma notation:
   (a) $1 + 4 + 7 + \ldots + 70$
   (b) $1 - 1 - 3 - \ldots - 111$
   (c) $1 + \frac{1}{2} + \frac{1}{4} + \ldots$
   (d) $2 - 6 + 18 - 54 + \ldots$ to $n$ terms

2. Calculate, by using an appropriate formula: $3 - 1 + \ldots + \ldots + \frac{1}{243}$

3. Evaluate:
   (a) $\sum_{r=0}^{49} (2r + 1)$
   (b) $\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^{k+2}$

4. Calculate $n$ if $\sum_{k=1}^{n} (3k + 1) = 650$

5. Determine the 60th term in the sequence $3; 8; 15; 24; \ldots$

6. Pascal’s triangle is represented below:

   
   \[
   \begin{array}{ccccccc}
   & & & & 1 & & \\
   & & & 1 & & 1 & \\
   & & 1 & & 2 & & 1 \\
   & 1 & & 3 & & 3 & & 1 \\
   1 & & 4 & & 6 & & 4 & & 1 \\
   \end{array}
   
   \]

   A sequence is formed by the sum of the digits of each line.
   (a) What type of sequence is formed?
   (b) What will the sum of the digits of Line 10 be?

7. 2; $x$; $y$ are the first three terms of an arithmetic sequence. If the 2nd term is decreased by 1, the three terms will form a geometric sequence. Calculate $x$ and $y$.

8. 12 and $\frac{3}{2}$ are the 1st and the 4th term of a particular sequence.
   (a) Determine the 2nd and 3rd terms if the sequence is arithmetic.
   (b) Determine the 2nd and 3rd terms if the sequence is geometric.
   (c) Determine the sum of the first ten terms if the sequence is geometric.

9. The 11th term of an arithmetic sequence is 32 and the sum of the first 11 terms is 187. Calculate the 21st term.

10. The 3rd and 12th terms of an arithmetic progression are 12 and $-24$ respectively. Determine the 49th term.

11. The 2nd term of a geometric series is 6 and the 5th term is 162. Find the sum of the first 20 terms.

12. The $n$th term of a geometric sequence is $8(5)^{n-2}$. Determine:
   (a) the first three terms.
   (b) the term of the sequence that has a value of 200.

13. Calculate the difference between $S_n$ and $S_r$ of $4 + \frac{4}{3} + \frac{4}{9} + \ldots$

14. For which values of $x$ will the series $2(3x - 1) + 2(3x - 1)^2 + 2(3x - 1)^3 + \ldots$ converge?
15. A rubber ball dropped from a height of 20 m loses 20% of its previous height on each rebound.
   (a) Calculate the height to which the ball will rise on the second rebound.
   (b) Determine the number of times it will rise to a height of more than 4m.
   (c) Show that the total distance the ball will travel before it comes to rest is 100 m.

16. The sum of the first 20 terms of the series \(2^x + 2^{x+1} + 3 \cdot 2^x + \ldots\) is 1680. Determine the value of \(x\).

17. In an arithmetic sequence, the sum of the first 8 terms is 88. The product of the 1st and the 8th terms is 120. Calculate values for the 1st term.

18. The sum to infinity of a convergent geometric series is 13,5. The sum to infinity of the same series calculated from the 3rd term onwards is 1,5. If \(r > 0\), determine the value of \(r\).

19. A square has each of its sides 1 unit and the midpoints of the sides are joined. Determine:
   (a) the length of each side of the new square.
   (b) the area of the new square.
   (c) the sum of the areas of all the squares if continued indefinitely.
   (d) the sum of all the perimeters of all the squares if continued indefinitely.

SOME CHALLENGES

1. The sum of a geometric series is 364 of which the first term is 243 and the last term is 1. Determine the constant ratio of the series.

2. Write the following series in sigma notation: \(x^0y + x^{-1}y + x^{1}y^2 + \ldots + x^{n}y^{n-1} + x^n y^n\)

3. What is the greatest value of \(p\) for which the series \(\sum_{k=1}^{6} 10(3)^{k-1} < 2300\)?

4. Consider the geometric sequence: \(8x^2, 4x^3, 2x^4, \ldots\)
   (a) Show that the general term is given by \(T_n = 2^{4-n} \cdot x^{1+n}\)
   (b) Calculate the sum of the series to infinity if \(x = \frac{3}{2}\).

5. Solve simultaneously for \(x\) and \(y\):

\[3y = 1 + x + x^2 \quad \text{and} \quad \sum_{k=3}^{4} x(k-2)^2 = 5y\]

6. Prove that: \(\sum_{k=3}^{n} (2k-1)n = n^3 - 4n\)

7. If \(ax^2 + 2bx + c = 0\) has equal roots, prove that \(a \; b \; c\) forms a geometric sequence.

8. If \(a^2 (c-b) \; a(c-b) \; b-a\) forms a geometric sequence, prove that \(a \; b \; c\) forms an arithmetic sequence.
9. Refer to the investigation at the end of the chapter before attempting the questions below.
   (a) The sum of \( n \) terms is given by \( S_n = \frac{n}{2}(1+n) \). Find the fifth term of the sequence.
   (b) The sum to \( n \) terms of an arithmetic series is \( S_n = \frac{n}{2}(7n+15) \).

10. If \( \sum_{m=1}^{5} (x-3m) = \sum_{m=1}^{8} (x-3m) \), prove that \( \sum_{r=1}^{13} (x-3m) = 0 \).

11. Show that \((x-1)(x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) = x^9 - 1\) without multiplying the terms out.

12. \( k \) is an even number in the series \( \frac{1}{k} + \frac{3}{k} + \frac{5}{k} + \ldots + \frac{k-1}{k} \).
   (a) Determine the number of terms in the series in terms of \( k \).
   (b) Determine the sum of the series in terms of \( k \).
   (c) Hence, or otherwise, evaluate:
       \[ \left( \frac{1}{4} + \frac{3}{4} \right) + \left( \frac{1}{6} + \frac{3}{6} + \frac{5}{6} \right) + \left( \frac{1}{8} + \frac{3}{8} + \frac{5}{8} + \frac{7}{8} \right) + \ldots + \left( \frac{1}{50} + \frac{3}{50} + \frac{5}{50} + \ldots + \frac{49}{50} \right) \]

13. Below is a 2-dimensional representation of the manner in which cans were stacked (you are seeing the bottom of each can). There are 26 cans in the bottom row. Determine the total number of cans that can be stacked.

   ![Cans Stacked Diagram]

INVESTIGATION

Consider the arithmetic series \( 2 + 5 + 8 + 11 + 14 + 17 + 20 + \ldots \) and answer the questions that follow. In this series, we define \( T_1 = S_1 = 2 \)

1. Determine the values of:
   (a) \( T_2 \), \( S_2 \) and \( S_1 \)
   (b) \( T_3 \), \( S_3 \) and \( S_2 \)
   (c) \( T_4 \), \( S_4 \) and \( S_3 \)

   What can you conclude?

2. Identify a relationship between \( T_n \), \( S_n \) and \( S_{n-1} \) where \( n > 1 \) and \( n \in \mathbb{N} \).

3. Hence determine the values of:
   (a) \( T_5 \) \hspace{1cm} (b) \( T_6 \)

4. In an arithmetic sequence \( S_n = n^2 - 2n \).
   Use your formula in 2 to determine the value of:
   (a) \( T_7 \) \hspace{1cm} (b) \( T_{50} \) \hspace{1cm} (c) \( T_n \)
CHAPTER 2 – INVERSE FUNCTIONS

REVISION OF THE CONCEPT OF A FUNCTION

We will briefly revise the concept of a function, which was studied in your Grade 10 year. A function is a rule by means of which each element of the domain is associated with only one element of the range. For a function, two or more elements of the domain may be associated with the same element of the range as well. However, a relation is not a function if one element of the domain is associated with more than one element of the range. Consider the following relations.

EXAMPLE 1

(a) \( \{(−2 ; 1) ; (4 ; 6) ; (5 ; 7) ; (3 ; 9)\} \)

Here, each element of the domain is associated with only one element of the range. In other words, each \( x \)-value associates with only one \( y \)-value. In this case, the relation is said to be a one-to-one function.

(b) \( \{(−2 ; 16) ; (0 ; 4) ; (1 ; 4) ; (3 ; 7)\} \)

Here, each element of the domain is associated with only one element of the range. However, the \( x \)-values 0 and 1 are associated with the same element of the range (namely 4). In this case, the relation is said to be a many-to-one function. Each \( x \)-value still associates with only one \( y \)-value.

(c) \( \{(−2 ; 16) ; (4 ; 1) ; (4 ; 6) ; (3 ; 7)\} \)

Here, the \( x \)-value 4 in the domain is associated with more than one element of the range (1 and 6). In this case, the relation is one-to-many and is not a function.

The Vertical and Horizontal Line Tests

We can use a ruler to perform the “vertical line test” on a graph to see whether it is a function or not. Hold a clear plastic ruler parallel to the \( y \)-axis, i.e. vertical. Move it from left to right over the axes. If the ruler only ever cuts the curve in one place
as the ruler moves from left to right across the graph, then the graph is a function. If the ruler at any stage cuts the graph in more than one place, then the graph is not a function. This is because the same $x$-value will be associated with more than one $y$-value.

The “horizontal line test” determines if the graph is a **one-to-one** or **many-to-one function**. If the ruler is positioned horizontally so that it is parallel to the $x$-axis, and the movement of the ruler is horizontally up or down, the following holds true: If the ruler only ever cuts the curve in one place as the ruler moves horizontally up or down across the graph, then the graph is a one-to-one function. If the ruler at any stage cuts the graph in more than one place, then the graph is not a one-to-one function, but rather a many-to-one function.

**EXAMPLE 2**

Determine whether the following relations are functions or not. If the graph is a function, determine whether the function is one-to-one or many-to-one.

(a) (b) 
\[
x \begin{array}{c}
0 \\
1 \\
2 \\
\end{array}
\begin{array}{c}
0 \\
1 \\
2 \\
\end{array}
\]
\[
y \begin{array}{c}
1 \\
2 \\
\end{array}
\begin{array}{c}
1 \\
2 \\
\end{array}
\]
\[
\text{one-to-one function}
\]

(c) (d) 
\[
x \begin{array}{c}
0 \\
1 \\
2 \\
\end{array}
\begin{array}{c}
0 \\
1 \\
2 \\
\end{array}
\]
\[
y \begin{array}{c}
1 \\
2 \\
\end{array}
\begin{array}{c}
1 \\
2 \\
\end{array}
\]
\[
\text{many-to-one function}
\]

(e) (f) 
\[
x \begin{array}{c}
0 \\
1 \\
2 \\
\end{array}
\begin{array}{c}
0 \\
1 \\
2 \\
\end{array}
\]
\[
y \begin{array}{c}
-1 \\
-2 \\
\end{array}
\begin{array}{c}
-1 \\
-2 \\
\end{array}
\]
\[
\text{one-to-one function}
\]

(g) 
\[
x \begin{array}{c}
0 \\
1 \\
2 \\
\end{array}
\begin{array}{c}
0 \\
1 \\
2 \\
\end{array}
\]
\[
y \begin{array}{c}
-1 \\
-2 \\
\end{array}
\begin{array}{c}
-1 \\
-2 \\
\end{array}
\]
\[
\text{not a function}
\]
EXERCISE 1

Determine whether each of the following relations is a function or not. If the relation is a function, determine whether it is one-to-one or many-to-one. Also write down the domain and range for each relation.

(a) $y = 2x + 3$
(b) $y = x^2$
(c) $y = \sqrt{x}$
(d) $y = \frac{1}{x}$
(e) $y = |x|$
(f) $y = \sin(x)$

(g) [Diagram of a one-to-one function]
(h) [Diagram of a not a function]
Mapping and functional notation

Since functions are special relations, we reserve certain notation strictly for use when dealing with functions. Consider the function \( f = \{(x, y) \mid y = 3x\} \).

This function may be represented by means of mapping notation or functional notation.

Mapping notation

\[ f : x \mapsto 3x \]

This is read as “\( f \) maps \( x \) onto \( 3x \)”. If \( x = 2 \) is an element of the domain, then the corresponding element in the range is \( 3(2) = 6 \). We say that 6 is the image of 2 in the mapping of \( f \).

Functional notation

\[ f(x) = 3x \]

This is read as “\( f \) of \( x \) is equal to \( 3x \)”.

The symbol \( f(x) \) is used to denote the element of the range to which \( x \) maps. In other words, the \( y \)-values corresponding to the \( x \)-values are given by \( f(x) \), i.e.

\[ y = f(x) \).

For example, if \( x = 4 \), then the corresponding \( y \)-value is obtained by substituting \( x = 4 \) into \( 3x \).

For \( x = 4 \), the \( y \)-value is \( f(4) = 3(4) = 12 \).

The brackets in the symbol \( f(4) \) do not mean \( f \) times 4, but rather the \( y \)-value at \( x = 4 \).
INVERSE FUNCTIONS

INVERSES OF ONE-TO-ONE LINEAR FUNCTIONS

EXAMPLE 3

Consider the function \( f(x) = 2x - 4 \).

It is clear that \( f \) is the rule that transforms given values in the domain (\( x \)-values) into values in the range (\( y \)-values) using the rule \( 2x - 4 \).

For example, if \( x = 3 \), then the function \( f \) transforms this \( x \)-value into a corresponding \( y \)-value in the range as follows:

\[
\begin{align*}
\text{original equation} & : \quad 2x - 4 = y \\
\text{plug } x = 3 & : \quad 2(3) - 4 = y \\
& \implies f(3) = 2 \\
\end{align*}
\]

\( \therefore y = 2 \)

So if \( x = 3 \), then \( y = 2 \).

The rule that reverses this process and transforms 2 back to 3 is called the inverse of the original function \( f(x) = 2x - 4 \). This rule is obtained by interchanging \( x \) and \( y \) in the equation \( y = 2x - 4 \) and then making \( y \) the subject of the formula.

The new function is called the inverse function of the original function \( f \) and is denoted by \( f^{-1} \).

So if \( y = 2x - 4 \) \( \quad (f) \)

Then \( x = 2y - 4 \) \( \quad (\text{interchange } x \text{ and } y) \)

\( \therefore 2y = x + 4 \)

\( \therefore y = \frac{x + 4}{2} \) \( \quad (f^{-1}) \)

Now if we let \( x = 2 \), then

\( y = \frac{2 + 4}{2} = 3 \)

So rule \( f \) transforms 3 into 2 and the reverse (or inverse rule) \( f^{-1} \) transforms 2 back into 3.

We define this inverse function as \( f^{-1}(x) = \frac{x + 4}{2} \).

<table>
<thead>
<tr>
<th>Original function</th>
<th>Inverse function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = 2x - 4 )</td>
<td>( f^{-1}(x) = \frac{x + 4}{2} )</td>
</tr>
<tr>
<td>( f ) transforms 3 into 2 as follows:</td>
<td>( f^{-1} ) transforms 2 back into 3 as follows:</td>
</tr>
<tr>
<td>( f(3) = 2(3) - 4 = 2 )</td>
<td>( f^{-1}(2) = \frac{2 + 4}{2} = 3 )</td>
</tr>
</tbody>
</table>
If we now draw the graphs of these two functions, it is clear that they are reflections of each other about the line \( y = x \).

\[ y = 2x - 4 \quad (f) \]
\[ y = 2(0) - 4 = -4 \]
\[ x\text{-intercept:} \quad 0 = 2x - 4 \]
\[ \therefore -2x = -4 \]
\[ \therefore x = 2 \]

\[ y = \frac{x + 4}{2} \quad (f^{-1}) \]
\[ y = \frac{0 + 4}{2} = 2 \]
\[ x\text{-intercept:} \quad 0 = \frac{x + 4}{2} \]
\[ \therefore 0 = x + 4 \]
\[ \therefore x = -4 \]

**Notice:**

- The coordinates of the point of intersection of these two graphs can be determined by solving the equation:
  \[ f(x) = f^{-1}(x) \]
  \[ \therefore 2x - 4 = \frac{x + 4}{2} \]
  \[ \therefore 4x - 8 = x + 4 \quad (\text{LCD} = 2) \]
  \[ \therefore 3x = 12 \]
  \[ \therefore x = 4 \]
  \[ \therefore y = 2(4) - 4 \]
  \[ \therefore y = 4 \]
  The coordinates of the point of intersection are \((4 ; 4)\)

- The function \( f(x) = 2x - 4 \) is a one-to-one linear function.
  Its inverse \( f^{-1}(x) = \frac{x + 4}{2} \) is also a one-to-one function.

- The two graphs are symmetrical to each other about the line \( y = x \).

**EXERCISE 2**

Consider the following functions:

(a) \( f(x) = 2x + 4 \)  
(b) \( f(x) = 3x - 6 \)  
(c) \( f(x) = \frac{1}{2}x + 3 \)  
(d) \( f(x) = 4x - 1 \)  
(e) \( f(x) = -2x \)  
(f) \( f(x) = 2x - 5 \)
(1) For each function determine \( f^{-1} \), the inverse function in the form \( f^{-1}(x) = \ldots \).
(2) Hence draw neat sketch graphs of both functions on the same set of axes.
(3) Draw the line of symmetry on the same set of axes as the two graphs.
(4) Determine the coordinates of the point of intersection of both graphs and then indicate this point on the diagram.

It should now be clear that the inverses of one-to-one functions are also functions for all real values of \( x \). However, the situation changes with many-to-one functions. With many-to-one functions, what happens is that the graph of the inverse relation is not a function. The only way to ensure that the inverse relation of a many-to-one function is itself a function is to restrict the domain of the original many-to-one function. This means that you have to restrict the domain so that the original many-to-one function now becomes a one-to-one function.

**INVERSES OF MANY-TO-ONE QUADRATIC FUNCTIONS**

**EXAMPLE 4**

Consider the many-to-one function \( f(x) = x^2 \).

We will now draw the graph of this function as well as its reflection about the line \( y = x \).

\[
y = x^2
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>-1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

\( x = y^2 \)  (Interchange \( x \) and \( y \) to get the inverse relation)

The \( y \)-values are now independent with the \( x \)-values dependent.

<table>
<thead>
<tr>
<th>( y )</th>
<th>-1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

It is possible to make \( y \) the subject of the formula for the inverse relation:

\[
\therefore y^2 = x
\]

\[
\therefore y = \pm \sqrt{x} \quad \text{provided } x \geq 0
\]

The graph of \( y = \pm \sqrt{x} \) is not a function because a vertical line will cut the graph in two points as it moves from left to right. So we will need to do something to the graph of \( y = x^2 \) so that when we determine the inverse, this inverse will also be a function.
We can do this in one of two ways:

**Situation 1**

Restrict the domain of $f(x) = x^2$ as follows:

$$f(x) = x^2 \text{ where } x \geq 0$$

Whenever the domain of the original function is restricted, it is important to ensure that the range of the restricted function remains the same as the original function.

In this situation, the range of the original function is $y \in [0, \infty)$ and the range of the restricted function is the same. By keeping the range the same for the restricted function, the inverse of this restricted function will also be a function.

It is clear that the inverse of the graph of the function $f(x) = x^2$ where $x \geq 0$ is also a function.

Notice too that both graphs are also one-to-one functions.

The equation of the inverse function is then defined as $f^{-1}(x) = \sqrt{x}$ where $x \geq 0$ and $y \geq 0$

**Situation 2**

Restrict the domain of $f(x) = x^2$ as follows:

$$f(x) = x^2 \text{ where } x \leq 0$$

Once again, the range of the restricted function is the same as the range of the original function.

It is clear that the inverse of the graph of the function $f(x) = x^2$ where $x \leq 0$ is also a function.

Notice too that both graphs are also one-to-one functions.

The equation of the inverse function is then defined as $f^{-1}(x) = -\sqrt{x}$ where $x \geq 0$ and $y \leq 0$
EXERCISE 3

1. Consider the following many-to-one functions given below:
   (a) \( f(x) = 2x^2 \)  
   (b) \( g(x) = 3x^2 \)  
   (c) \( p(x) = -x^2 \)
   (d) \( f(x) = -2x^2 \)  
   (e) \( g(x) = \frac{1}{2}x^2 \)  
   (f) \( p(x) = -\frac{1}{2}x^2 \)

   (1) Sketch the graph of each function and its inverse relation on the same set of axes.
   (2) Now restrict the domain of the original function in two different ways so as to form new one-to-one functions.
   (3) Sketch the graphs of each new function and its inverse function on the same set of axes. Use a separate set of axes for each function and its inverse.
   (4) Hence rewrite the equation of each inverse function in the form \( f^{-1}(x) = \ldots \)
   (5) Write down the domain and range for each graph drawn.

2. The diagram below represents the graphs of the functions \( f \) and \( g \). The two graphs intersect at A.
   (a) Determine the coordinates of A.
   (b) Determine the equation of the inverse of the function \( f \) in the form \( f^{-1}(x) = \ldots \)
   (c) Redraw the diagram and then sketch the graphs of the following on your diagram:
      (1) the reflection of \( f \) about the line \( y = x \). Write down the equation of this function.
      (2) the reflection of \( g \) about the y-axis. Write down the equation of this function.
      (3) the reflection of \( f \) about the x-axis. Write down the equation of this function.

THE INVERSE OF THE EXPONENTIAL FUNCTION

EXAMPLE 5

Consider the function \( f(x) = 2^x \).
Since this function is a one-to-one exponential function, its inverse will also be a one-to-one function. The domain will not need to be restricted in any way.
We will now sketch the graph of this function and its inverse on the same set of axes.

\[
y = 2^x \quad (f) \\
\begin{array}{c|c|c|c}
x & -1 & 0 & 1 \\
2^x & \frac{1}{2} & 1 & 2 \\
\end{array}
\]
A problem now arises if you want to make \( y \) the subject of the formula in the equation \( x = 2^y \), which cannot be done by using any of the methods you have studied thus far. A Scottish mathematician named John Napier (1550-1617) devised a clever way of making \( y \) the subject of the formula. He introduced a notation referred to as a logarithm. We will now discuss the concept of a logarithm and then later on develop the theory of logarithms in more detail.

If a number is written in exponential form, then the exponent is called the **logarithm** of the number. For example, the number 8 can be written in exponential form as \( 8 = 2^3 \). Clearly, the exponent in this example is 3 and the base is 2. We can then say that the logarithm of 8 to base 2 is 3. This can be written as \( \log_2 8 = 3 \).

The base 2 is written as a sub-script between the “log” and the number 8.

In general, we can rewrite

number = base\textsuperscript{exponent} in logarithmic form as follows:

\[
\log_{\text{base}} \text{(number)} = \text{exponent}
\]

**EXAMPLE 6**

(a) Express the following in logarithmic form:

(1) \[ 16 = 4^2 \]
\[ \therefore \log_4 16 = 2 \]

(2) \[ \sqrt{3} = 3^{\frac{1}{2}} \]
\[ \therefore \log_3 \sqrt{3} = \frac{1}{2} \]

(3) \[ 10^3 = 1000 \]
\[ \therefore 1000 = 10^3 \]
\[ \therefore \log_{10} 1000 = 3 \]

(b) Express the following in exponential form:

(1) \[ \log_3 9 = 2 \]
\[ \therefore 9 = 3^2 \]

(2) \[ \log_2 64 = 6 \]
\[ \therefore 64 = 2^6 \]

(3) \[ \frac{1}{2} = \log_5 \sqrt{5} \]
\[ \therefore \log_5 \sqrt{5} = \frac{1}{2} \]
\[ \therefore 5^{\frac{1}{2}} = \sqrt{5} \]

(4) \[ \log_3 y = x \]
\[ \therefore y = 3^x \]
Note:
It is now possible to make $y$ the subject of the formula in the equation $x = 2^y$ by means of the concept of a logarithm.

If $x = 2^y$, then it is clear from the definition of a logarithm that $\log_2 x = y$. In other words, we can write the inverse of the function $f(x) = 2^x$ as $f^{-1}(x) = \log_2 x$. We call this inverse of the exponential function a logarithmic function.

**EXAMPLE 7**

Consider the function $g(x) = \left(\frac{1}{2}\right)^x$.

Since this function is a one-to-one exponential function, its inverse will also be a one-to-one function. The domain will not need to be restricted in any way.

We will now sketch the graph of this function and its inverse (logarithmic graph) on the same set of axes.

\[
\begin{array}{c|c|c|c}
  x & -1 & 0 & 1 \\
  \left(\frac{1}{2}\right)^x & 2 & 1 & \frac{1}{2} \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
  y & -1 & 0 & 1 \\
  \left(\frac{1}{2}\right)^y & 2 & 1 & \frac{1}{2} \\
\end{array}
\]

You can now make $y$ the subject of the formula by using the logarithm concept:

\[
x = \left(\frac{1}{2}\right)^y
\]

∴ $\log_{\frac{1}{2}} x = y$

i.e. $g^{-1}(x) = \log_{\frac{1}{2}} x$
Summary of the two types of logarithmic functions

You will have probably noticed that there are two types of logarithmic functions which are the inverses of the corresponding exponential functions. If the base is a fraction between 0 and 1, the logarithmic graph is decreasing. If the base is greater than 1, then the logarithmic graph will increase. The domain for these functions is restricted to all real values of $x$ which are positive, i.e. $x > 0$.

Some important theory emerges from these graphs:
The expression $\log_a x$ is defined only if:
- $0 < a < 1$ or $a > 1$
  (a cannot be negative, zero or 1)
- $x > 0$
  (x cannot be negative or zero)

EXAMPLE 8

The function $f: x \rightarrow \log_a x$ passes through the point $(4 ; -2)$.
(a) Determine the value of $a$ and hence the equation of $f$.
(b) Write down the equation of the inverse of $f$ in the form $f^{-1}(x) = ....$
(c) Draw neat sketch graphs of $f$ and $f^{-1}$ on the same set of axes.
(d) Write down the domain and range of each function.
(e) Determine the values of $x$ for which the functions are either increasing or decreasing.

Solutions

(a) $y = \log_a x$ passes through the point $(4 ; -2)$
$-2 = \log_a 4$
∴ $a^{-2} = 4$
∴ $\frac{1}{a^2} = 4$
∴ $1 = 4a^2$
∴ $4a^2 = 1$
∴ $a^2 = \frac{1}{4}$
But since $a$ cannot be negative,
∴ $a = \pm \frac{1}{2}$
∴ $f(x) = \log_{\frac{1}{2}} x$
$\therefore a = \frac{1}{2}$
(b) \[ y = \log_{\frac{1}{2}} x \quad (f) \]
\[ \therefore x = \log_{\frac{1}{2}} y \quad (f^{-1}) \]
\[ \therefore \left(\frac{1}{2}\right)^x = y \]
\[ \therefore f^{-1}(x) = \left(\frac{1}{2}\right)^x \]

(c) To sketch the graph of \( y = \log_{\frac{1}{2}} x \), first change to exponential form and then use the table method.
\[ y = \log_{\frac{1}{2}} x \quad (f) \]
\[ \therefore \left(\frac{1}{2}\right)^y = x \]
\[ \therefore x = \left(\frac{1}{2}\right)^y \]

To sketch the graph of \( y = \left(\frac{1}{2}\right)^x \), use a table.

\[
\begin{array}{c|c|c|c}
 x & -1 & 0 & 1 \\
\hline
\left(\frac{1}{2}\right)^x & 2 & 1 & \frac{1}{2} \\
\end{array}
\]

Now sketch the graphs of the two functions on the same set of axes.

(d) Domain of \( f \): \( x \in (0; \infty) \)
Domain of \( f^{-1} \): \( x \in (-\infty; \infty) \)
Range of \( f \): \( y \in (-\infty; \infty) \)
Range of \( f^{-1} \): \( y \in (0; \infty) \)

(e) Both functions are decreasing:
The graph of \( f \) decreases for the interval \( x \in (0; \infty) \)
The graph of \( f^{-1} \) decreases for the interval \( x \in (-\infty; \infty) \)
EXAMPLE 9

Determine the domain of the function \( f(x) = \log_2(1 - 2x) \).

Solution

From the important theory on logarithms, it is clear that for this expression to be defined, the expression \( 1 - 2x \) must be strictly positive.

\[
1 - 2x > 0 \\
\therefore x < \frac{1}{2}
\]

The domain is therefore \( x \in \left(-\infty; \frac{1}{2}\right) \).

EXERCISE 4

1. For each function \( f \) below, write down the equation of the inverse function in the form \( f^{-1}(x) = \ldots \) and hence sketch both graphs on the same set of axes.

   (a) \( f(x) = 5^x \) \hspace{1cm} (b) \( f(x) = \left(\frac{1}{5}\right)^x \)

   (c) \( f(x) = \log_4 x \) \hspace{1cm} (d) \( f(x) = \log_\frac{1}{4} x \)

2. The graph of \( f : x \rightarrow a^x \) passes through the point \( \left(2; \frac{9}{4}\right) \).

   (a) Calculate the value of \( a \).

   (b) Write down the equation of the inverse in the form \( f^{-1}(x) = \ldots \).

   (c) Determine the equation of \( g \), the reflection of \( f \) about the y-axis.

   (d) Determine the equation of \( h \), the reflection of \( f \) about the x-axis.

   (e) Sketch the graphs of \( f, g, h \) and \( f^{-1} \) on the same set of axes.

   (f) Write down the domain and range of \( f \) and \( g \).

3. The graph of \( f : x \rightarrow a^x \) passes through the point \( (-2; 4) \).

   (a) Calculate the value of \( a \).

   (b) Write down the equation of the inverse in the form \( f^{-1}(x) = \ldots \).

   (c) Determine the equation of \( g \), the reflection of \( f \) about the y-axis.

   (d) Determine the equation of \( h \), the reflection of \( f \) about the x-axis.

   (e) Sketch the graphs of \( f, g, h \) and \( f^{-1} \) on the same set of axes.

   (f) Write down the domain and range of \( f \) and \( g \).

4. The graph of \( f : x \rightarrow \log_a x \) passes through the point \( (16; 2) \).

   (a) Calculate the value of \( a \).

   (b) Write down the equation of the inverse in the form \( f^{-1}(x) = \ldots \).

   (c) Sketch the graphs of \( f \) and \( f^{-1} \) on the same set of axes.

5. Determine the domain of the following functions:

   (a) \( f(x) = \log_3(3x - 1) \) \hspace{1cm} (b) \( f(x) = \log_4(3 - 5x) \)

   (c) \( f(x) = \log_2(x^2 - 4) \) \hspace{1cm} (d) \( f(x) = \log_x(x^2 + 4) \)
THE THEORY OF LOGARITHMS

As already mentioned, the idea of a logarithm was invented by the famous Scottish mathematician, John Napier. Due to his invention, a whole new branch of mathematics emerged, which has some wonderful applications in the real world. A number of laws were discovered and mathematics was once again taken to a higher level. A classical example of the use of logarithms is to be found in solving exponential equations where the bases are not the same. You probably recall having used trial and error to solve equations such as \(3^x = 5\). The use of logarithms is a far more effective tool for solving these types of equations. We will now explore the beauty of the laws of logarithms.

**LAW 1**
\[
\log_a (x \times y) = \log_a x + \log_a y \quad (x > 0, \ y > 0, \ 0 < a < 1 \text{ or } a > 1)
\]

**Proof** *(not for examination purposes)*

If \(\log_a x = P\) then \(x = a^P\)

If \(\log_a y = Q\) then \(y = a^Q\)

If \(\log_a (x \times y) = R\) then \(x \times y = a^R\)

Now since \(xy = a^R\), it is clear that \(a^P \cdot a^Q = a^R\)

\[\therefore a^{P+Q} = a^R\]

\[\therefore P + Q = R\]

\[\therefore \log_a x + \log_a y = \log_a (x \times y)\]

or \(\log_a (x \times y) = \log_a x + \log_a y\)

**LAW 2**
\[
\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y \quad (x > 0, \ y > 0, \ 0 < a < 1 \text{ or } a > 1)
\]

**Proof** *(not for examination purposes)*

If \(\log_a x = P\) then \(x = a^P\)

If \(\log_a y = Q\) then \(y = a^Q\)

If \(\log_a \left(\frac{x}{y}\right) = R\) then \(\frac{x}{y} = a^R\)

Now since \(\frac{x}{y} = a^R\), it is clear that \(\frac{a^P}{a^Q} = a^R\)

\[\therefore a^{P-Q} = a^R\]

\[\therefore P - Q = R\]

\[\therefore \log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)\]

or \(\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y\)
LAW 3
\[ \log_a x^m = m \log_a x \quad (x > 0, \ y > 0, \ 0 < a < 1 \ or \ a > 1) \]

Proof (not for examination purposes)
If \( \log_a x^m = P \) then \( x^m = a^P \) ..........A
If \( \log_a x = Q \) then \( x = a^Q \) ..........B

Now substitute B into A:
\[ x^m = a^P \]
\[ \therefore (a^Q)^m = a^P \]
\[ \therefore a^{Qm} = a^P \]
\[ \therefore Qm = P \]
\[ \therefore \log_a x^m = \log_a x \]

EXAMPLE 10

(a) Expand the following using the laws of logarithms:
1. \( \log_3 5x \)
2. \( \log_4 \left( \frac{5}{x} \right) \)
3. \( \log_7 (x^4y) \)
4. \( \log_3 \left( \frac{4a^2}{bc} \right) \)

(b) Rewrite \( 2 \log_3 a + \log_3 b - \log_3 c \) as the logarithm of a single expression.

Solutions

(a) (1) \( \log_3 5x = \log_3 5 + \log_3 x \) (LAW 1)
(2) \( \log_4 \left( \frac{5}{x} \right) = \log_4 5 - \log_4 x \) (LAW 2)
(3) \( \log_7 (x^4y) = \log_7 x^4 + \log_7 y \) (LAW 1)
\[ \therefore \log_7 (x^4y) = 4 \log_7 x + \log_7 y \] (LAW 3)
(4) \( \log_3 \left( \frac{4a^2}{bc} \right) = \log_3 (4a^2) - \log_3 (bc) \) (LAW 2)
\[ \therefore \log_3 \left( \frac{4a^2}{bc} \right) = \log_3 4 + \log_3 a^2 - (\log_3 b + \log_3 c) \] (LAW 1)
\[ \therefore \log_3 \left( \frac{4a^2}{bc} \right) = \log_3 4 + 2 \log_3 a - \log_3 b - \log_3 c \] (LAW 3)

(b) \( 2 \log_3 a + \log_3 b - \log_3 c \)
\[ = \log_3 a^2 + \log_3 b - \log_3 c \] (LAW 3)
\[ = \log_3 (a^2b) - \log_3 c \] (LAW 1)
\[ = \log_3 \left( \frac{a^2b}{c} \right) \] (LAW 2)
Note:
Take note of the following exceptions to the laws of logarithms:
\[ \log_a (x + y) \neq \log_a x + \log_a y \]
\[ \log_a (x - y) \neq \log_a x - \log_a y \]
\[ (\log_a x)^m = m \log_a x \]

EXERCISE 5
1. Expand the following expressions using the laws of logarithms:
   (a) \( \log_8 3x \)  
   (b) \( \log_8 \left( \frac{6}{x} \right) \)  
   (c) \( \log_9 x^4 \)  
   (d) \( \log_3 (xy^2) \)  
   (e) \( \log_4 (a^2b^5) \)  
   (f) \( \log_5 \left( \frac{2m^2}{n} \right) \)  
   (g) \( \log_2 \left( \frac{5a^3}{b^2c} \right) \)  
   (h) \( \log_m \left( \frac{(x + y)^2}{x} \right) \)

2. Rewrite the following as the logarithm of a single expression:
   (a) \( \log_4 x + \log_4 y \)  
   (b) \( \log_4 x - \log_4 y \)  
   (c) \( 3 \log_4 x \)  
   (d) \( 2 \log_b a - 3 \log_b c \)  
   (e) \( \log_3 (x + y) + \log_3 x - \log_3 y \)  
   (f) \( \log_2 3 - 2 \log_2 x + 2 \log_2 y \)

DEDUCTIONS BASED ON THE LAWS OF LOGARITHMS

DEDUCTION 1 \( \log_a a = 1 \) \hspace{1cm} (since \( a = a^1 \))

DEDUCTION 2 \( \log_a 1 = 0 \) \hspace{1cm} (since \( 1 = a^0 \))

Note: A logarithm to base 10 is called the common logarithm.
This logarithm is written without showing the base 10.
In other words, \( \log_{10} x \) is written simply as \( \log x \).

EXAMPLE 11
Simplify the following by using the laws and deductions of logarithms:

(a) \( \log_9 27 + \log_9 3 \)
   \[ = \log_9 (27 \times 3) \] \hspace{1cm} (LAW 1)
   \[ = \log_9 81 \]
   \[ = \log_9 9^2 \]
   \[ = 2 \log_9 9 \] \hspace{1cm} (LAW 3)
   \[ = 2 \times 1 \] \hspace{1cm} (DEDUCTION 1)
   \[ = 2 \]
(b) \[ \log_3 27 - \log_3 3 = \log_3 \left( \frac{27}{3} \right) \quad \text{(LAW 2)} \]
\[ = \log_3 9 \]
\[ = \log_3 3^2 \]
\[ = 2 \log_3 3 \quad \text{(LAW 3)} \]
\[ = 2 \times 1 \quad \text{(DEDUCTION 1)} \]
\[ = 2 \]

(c) \[ \log 20 + \log 30 - \log 6 + \log_4 1 \]
\[ = \log 20 + \log 30 - \log 6 + 0 \quad \text{(DEDUCTION 2)} \]
\[ = \log(20 \times 30) - \log 6 \quad \text{(LAW 1)} \]
\[ = \log 600 - \log 6 \]
\[ = \log \left( \frac{600}{6} \right) \quad \text{(LAW 2)} \]
\[ = \log 100 \]
\[ = \log_{10} 10^2 \]
\[ = 2 \log_{10} 10 \quad \text{(LAW 3)} \]
\[ = 2 \times 1 \quad \text{(DEDUCTION 1)} \]
\[ = 2 \]

(d) \[- \log 4 - \log 25 \]
\[ = -(\log 4 + \log 25) \]
\[ = -\log(4 \times 25) \]
\[ = -\log 100 \]
\[ = -\log_{10} 10^2 \]
\[ = -2 \log_{10} 10 = -2 \times 1 = -2 \]

(e) \[ \log_3 \frac{1}{27} \]
\[ = \log_3 27^{-1} \]
\[ = -\log_3 27 \]
\[ = -\frac{1}{2} \log_3 27 \]
\[ = -\frac{1}{2} \log_3 3^3 \]
\[ = -\frac{3}{2} \log_3 3 \]
\[ = \frac{3}{2} \]

(f) \[ \log_3 \sqrt{27} \]
\[ = \log_3 27^{\frac{1}{2}} \]
\[ = \frac{1}{2} \log_3 27 \]
\[ = \frac{1}{2} \log_3 3^3 \]
\[ = \frac{3}{2} \log_3 3 \]

(g) \[ \frac{(\log_2 8)^2}{\log_2 8^2} \]
\[ = \frac{(\log_2 2^3)^2}{2 \log_2 8} \]
\[ = \frac{(3 \log_2 2)^2}{2 \log_2 2^3} \]
\[ = \frac{(3 \times 1)^2}{2 \times 3 \log_2 2} \]
\[ = \frac{9}{6} \]
\[ = \frac{3}{2} \]

(h) \[ \frac{\log 16}{\log 4} \]
\[ = \frac{\log 2^4}{\log 2^2} \]
\[ = \frac{4 \log 2}{2 \log 2} \]
\[ = 2 \]
EXERCISE 6

Simplify the following by using the laws of logarithms:

(a) \( \log 2 + \log 5 \)
(b) \( \log_3 12 - \log_3 4 \)
(c) \( \log_2 32 - \log_2 4 - \log_2 1 \)
(d) \( \log_7 7 - \log_7 49 \)
(e) \( 2\log 5 + 3\log 2 - \frac{1}{3}\log 8 \)
(f) \( \log_6 15 + \log_6 3 - \log_6 5 + \log_6 4 \)
(g) \( \log_3 15 - \log_3 10 + \log_3 18 \)
(h) \( -\log 50 - \log 20 \)
(i) \( \log_3 81 - \log_3 \frac{1}{9} \)
(j) \( \log_3 81 + \log_3 \frac{1}{9} \)
(k) \( \log_2 128 + \log_2 8 \)
(l) \( \log_2 128 + \log_2 8 \)
(m) \( \frac{\log_5 125 - \log_5 25}{\log(125 - 25)} \)
(n) \( \frac{\log 27}{\log 3} + (\log_3 27)^2 \)
(o) \( (\log 0.01)^3 + \log 0.01^3 \)
(p) \( \log_2 \sqrt{32} \)
(q) \( 2\log_2 3 + 2 - \log_2 18 \)
(r) \( \frac{1}{2}\log_5 25 - \log_3 27 + \log_4 16 \)
(s) \( \frac{\log 9 - \log 3}{\log 9 + \log 3} \)
(t) \( \frac{\log_5 a^2 + 2}{\log_5 (5a)} \)

LAW 4 (Change of base law)

\[ \log_a x = \frac{\log_b x}{\log_b a} \]

**Proof** (not for examination purposes)

Let \( \log_b x = P \) and \( \log_b a = Q \), \( \log_a x = R \)

\[ \therefore x = b^P \]
\[ \therefore a = b^Q \]
\[ \therefore x = a^R \]

Now since \( x = b^P \) and \( x = a^R \)

\[ \therefore b^P = a^R \]

But since \( a = b^Q \)

\[ \therefore b^P = (b^Q)^R \]
\[ \therefore b^P = b^{QR} \]

\[ \therefore P = QR \]

\[ \therefore \log_b x = (\log_b a)(\log_a x) \]

\[ \therefore \frac{\log_b x}{\log_b a} = \log_a x \]

EXAMPLE 12

Simplify the following using the laws of logarithms:

(a) \( \log_9 27 \)
(b) \( \log_3 2 \cdot \log_2 5 \cdot \log_5 9 \)

**Solutions**

(a) There are two methods for simplifying this expression:
Method 1  
Change to base 3  
\[ \log_9 27 \]
\[ = \frac{\log_3 27}{\log_3 9} \]
\[ = \frac{\log_3 3^3}{\log_3 3^2} \]
\[ = \frac{3 \log_3 3}{2 \log_3 3} \]
\[ = \frac{3}{2} \]

Method 2  
Change to base 10  
\[ \log_9 27 \]
\[ = \frac{\log_{10} 27}{\log_{10} 9} \]
\[ = \frac{\log 27}{\log 9} \]
\[ = \frac{\log 3^3}{\log 3^2} \]
\[ = \frac{3 \log 3}{2 \log 3} \]
\[ = \frac{3}{2} \]

(b)  
\[ \log_2 3 \cdot \log_5 5 \cdot \log_9 9 \]
\[ = \frac{\log 2}{\log 3} \cdot \frac{\log 5}{\log 2} \cdot \frac{\log 9}{\log 5} \]
\[ = \frac{\log 2}{\log 3} \cdot \frac{\log 5}{\log 2} \cdot \frac{\log 3^2}{\log 5} \]
\[ = \frac{\log 2}{\log 3} \cdot \frac{\log 5}{\log 2} \cdot \frac{2 \log 3}{\log 5} \]
\[ = 2 \]

**EXERCISE 7**

Simplify the following by using the laws of logarithms:
(a)  \[ \log_8 16 + \log_9 27 \]
(b)  \[ \log_2 3 + \log_{25} 5 \]
(c)  \[ \log_3 7 \cdot \log_5 3 \cdot \log_7 25 \]
(d)  \[ \log_3 16 \cdot \log_5 9 \cdot \log_4 \frac{1}{5} \]

**SOLVING EQUATIONS USING LOGARITHMS**

**EXAMPLE 13**

Solve the following equation and round your answer off to three decimal places:
\[ 2^x = 3 \]

**Solution**

In Grade 10, you used the method of trial and error to solve this equation. However, you will probably agree that this method is quite cumbersome. The use of logarithms provides a far better way of solving exponential equations which normally would have had to be solved by the method of trial and error.

In order to solve for \( x \), simply apply the definition:
\[ 2^x = 3 \]
\[ \therefore x = \log_2 3 \]
\[ \therefore x = 1.585 \]
You used your calculator to calculate the value of $\log_2 3$. [Use the button $\log$]

Alternatively, if your calculator doesn’t have the button $\log$, you can use the following approach:

Take logarithms to base 10 on both sides of the equation as follows:

$2^x = 3$

$\therefore \log 2^x = \log 3$

$\therefore x \log 2 = \log 3$  \hspace{1cm} \text{(Law 3)}

$\therefore x = \frac{\log 3}{\log 2}$  \hspace{1cm} \text{(Divide both sides by $\log 2$)}

$\therefore x = 1.585$  \hspace{1cm} \text{(Use a calculator)}

Before discussing the next example, it is necessary to remind you about the restrictions for the logarithmic expression $\log_b a$.

The expression $\log_b a$ is defined for $a > 0$, $0 < b < 1$ or $b > 1$.

In other words, $a$ cannot be negative or zero, whereas the base $b$ cannot be negative, zero or 1.

**EXAMPLE 14**

Solve the following equations:

(a) $\log_5 x = 2$  \hspace{1cm} (b) $\log_x 4 = 2$

**Solutions**

(a) $\log_5 x = 2$

$\therefore x = 5^2$

$\therefore x = 25$

(b) $\log_x 4 = 2$

$\therefore 4 = x^2$

$\therefore x = \pm 2$

$\therefore x = 2$  \hspace{1cm} \text{(since $x \neq -2$)}

**EXERCISE 8**

1. Solve the following equations rounded off to three decimal places:

(a)  \hspace{1cm} (b)  \hspace{1cm} (c)  \hspace{1cm} (d)  \hspace{1cm} (e)  \hspace{1cm} (f)  \hspace{1cm} (g)  \hspace{1cm} (h)

$3^x = 4$  \hspace{1cm} $4^{2x} = 5$  \hspace{1cm} $3^{2x+1} = 12$  \hspace{1cm} $3 \cdot (2^x) = 0.3$  \hspace{1cm} $x = \log_4 5$  \hspace{1cm} $30(1,015)^x = 60$  \hspace{1cm} $3 \left( \frac{1}{3} \right)^x = 3 \cdot \frac{1}{3}$

$2x \cdot 7^x = \log 14$

2. Solve the following equations:

(a)  \hspace{1cm} (b)  \hspace{1cm} (c)  \hspace{1cm} (d)  \hspace{1cm} (e)  \hspace{1cm} (f)  \hspace{1cm} (g)  \hspace{1cm} (h)  \hspace{1cm} (i)  \hspace{1cm} (j)  \hspace{1cm} (k)  \hspace{1cm} (l)

$\log_2 x = 3$  \hspace{1cm} $\log_3 x = 2$  \hspace{1cm} $\log x = 0$  \hspace{1cm} $\log x = 1$  \hspace{1cm} $2 \log x = 8$  \hspace{1cm} $x = \log_2 32$  \hspace{1cm} $\log_2 \frac{1}{64} = 3x$  \hspace{1cm} $\log_x 4 = 2$  \hspace{1cm} $\log x 16 = 2$  \hspace{1cm} $\log x 8 = 3$  \hspace{1cm} $2 \log x = \log 25$  \hspace{1cm} $3 \log x = \log 27$
APPLICATIONS OF LOGARITHMS IN THE REAL WORLD

EXAMPLE 15

It has been established that there are new strains of bacteria which are immune to modern day antibiotics. This could have dreadful implications for the human race, particularly if these dangerous so called “super-bugs” are not able to be treated by modern medicine. In an experiment on a particular strain of these “super-bugs”, a scientist determined that the number of bacteria \( N \) in a culture increases exponentially after \( t \) days according to the formula \( N = 50 \cdot 2^t \).

(a) Determine how many “super-bugs” were put into the culture initially.
(b) Determine how many “super-bugs” were in the culture after 5 days.
(c) After how many days will there be 10,000 “super-bugs” in the culture?

Solutions

(a) At \( t = 0 \),

\[ N = 50 \cdot 2^0 = 50 \text{ “super-bugs”} \]

(b) At \( t = 5 \),

\[ N = 50 \cdot 2^5 = 1600 \text{ “super-bugs”} \]

(c) Here \( N = 10,000 \)

\[ N = 50 \cdot 2^t \]

\[ \therefore 10,000 = 50 \cdot 2^t \]

\[ \therefore 200 = 2^t \]

\[ \therefore \log 200 = \log 2^t \]

\[ \therefore \log 200 = t \log 2 \]

\[ \therefore \log 200 \]

\[ \therefore t = 8 \text{ days} \]

EXERCISE 9

1. A colony of an endangered species originally numbering 1000 was predicted to have a population \( N \) after \( t \) years given by the equation \( N = 1000(0.9)^t \).

(a) Estimate the population after 1 year.
(b) After how many years will the population decrease to 200?

2. Certain prescribed medicine that enters the human body is eventually eliminated by the body through urination. For an initial dose of 30 mg, suppose that the amount \( A \) remaining in the body \( n \) hours later is given by the formula \( A = 30(0.7)^n \).

(a) Estimate the amount of the medicine in the body 8 hours after the initial dose.
(b) What percentage of the medicine still in the body is eliminated each hour?
(c) How long will it take for the body to still have half the initial dosage?
3. The number of bacteria in a certain culture increased from 600 to 1800 between 08h00 and 10h00. The number (N) of bacteria t hours after 08h00 is given by the formula \( N = 600(3)^t \).

(a) Estimate the number of bacteria at 09h00, 11h00 and noon.
(b) How long will it take for the number of bacteria to reach 16200?

4. Paleontologists use a specific formula when determining the age of fossils.

This formula is the carbon-dating formula and is given by: \( P = \left( \frac{1}{2} \right)^{\frac{n}{5700}} \)

where \( P \) is the percentage of carbon-14 remaining in the fossils after \( n \) years. Calculate the approximate age of a certain fossil discovered if the percentage of carbon-14 in the fossil is 12.5%.

5. A water plant grows on the surface of a dam. The surface area covered by the plant doubles every day. At the beginning of a certain day, the plant covered one square metre of the surface.

(a) Complete the following table:

<table>
<thead>
<tr>
<th>Time ( t ) (in days)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area ( y ) covered in square metres</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Sketch the graph of the area \( y \) against time \( t \). Indicate the coordinates of the \( y \) – intercept and one other point on the curve.
(c) What is the equation of this graph?
(d) Determine the day on which the area reached 200 square meters.

**REVISION EXERCISE**

1. Consider the functions: \( f(x) = 2x^2 \) and \( g(x) = \left( \frac{1}{2} \right)^x \)

(a) Restrict the domain of \( f \) in one specific way so that the inverse of \( f \) will also be a function.
(b) Hence draw the graph of your new function \( f \) and its inverse function \( f^{-1} \) on the same set of axes.
(c) Write the inverse of \( g \) in the form \( g^{-1}(x) = .... \)
(d) Sketch the graph of \( g^{-1} \).
(e) Determine graphically the values of \( x \) for which \( \log_{\frac{1}{2}} x < 0 \)

2. The graph of \( f(x) = a^x \) passes through the point \( \left(-1; \frac{1}{2}\right) \).

(a) Calculate the value of \( a \).
(b) Write down the equation of the inverse of \( f \) in the form \( f^{-1}(x) = .... \)
(c) Sketch the graph of \( f \) and its inverse on the same set of axes.
(d) Write down the domain of \( f^{-1} \)
(e) Sketch the graph of \( y = f(-x) \) on the same set of axes.
3. The graph of \( f(x) = a^{x-1} \) passes through the point \( \left( 0; \frac{1}{2} \right) \).
   (a) Calculate the value of \( a \).
   (b) Write down the equation of the inverse of \( f \) in the form \( f^{-1}(x) = \ldots \).
   (c) Sketch the graph of \( f \) and its inverse on the same set of axes.
   (d) Write down the domain of \( f^{-1} \).

4. Sketched below are the graphs of \( f(x) = 3^x \) and \( g(x) = -x^2 \).
   (a) Write down the equation of the inverse of the graph of \( f(x) = 3^x \) in the form \( f^{-1}(x) = \ldots \).
   (b) On a set of axes, draw the graph of the inverse of \( f(x) = 3^x \).
   (c) Write down the domain of the graph of \( y = f^{-1}(x) \).
   (d) Explain why the inverse of the graph of \( y = g(x) = -x^2 \) is not a function.
   (e) Consider the graph of \( g(x) = -x^2 \).
      (1) Write down a possible restriction for the domain of \( g(x) = -x^2 \) so that the inverse of the graph of \( g \) will now be a function.
      (2) Hence draw the graph of the inverse function in (e)(1).
   (f) Explain how, using the transformation of the graph of \( f \), you would sketch the graphs of:
      (1) \( h(x) = -\log_3 x \)
      (2) \( p(x) = \left( \frac{1}{3} \right)^x + 1 \)

5. \( P(8;1\frac{1}{2}) \) is a point on the graph of \( f(x) = \log_a x \).
   (a) Show that \( a = 4 \).
   (b) Determine the equation of \( f^{-1} \) in the form \( y = \ldots \).
   (c) Determine the equation of \( g \) the reflection of \( f \) about the \( x \)-axis.
   (d) Sketch the graphs of \( f, g \) and \( f^{-1} \) on the same set of axes.

**SOME CHALLENGES**

1. Consider the graph of \( f \).
   (a) Is the graph of \( f \) a one-to-one function? Explain.
   (b) Write down the range of \( f \).
   (c) On a set of axes, draw the graph of the inverse of \( f \).
   (d) Explain why the inverse of \( f \) is not a function.

2. Calculate the sum of the following:
   (a) \( \log 2 + \log 4 + \log 8 + \ldots \) to 25 terms
   (b) \( \log 2 + \log 4 + \log 16 + \ldots \) to 15 terms
CHAPTER 3 – FINANCIAL MATHEMATICS

The revision exercise which follows is based on the work you did in Financial Mathematics during Grade 11. It is important that you master this exercise before moving on with the Grade 12 Financial Mathematics. Here is a summary of the formulae you dealt with in Grade 10 and 11:

Simple Interest: \[ A = P(1 + in) \]
Compound Interest: \[ A = P(1 + i)^n \] (Given P and A must be calculated)
\[ P = A(1 + i)^{-n} \] (Given A and P must be calculated)
Depreciation:
\[ A = P(1 - in) \] (Linear or straight-line depreciation)
\[ A = P(1 - i)^n \] (Reducing-balance depreciation)
\[ i_{\text{eff}} = \left(1 + \frac{i_{\text{nom}}}{n}\right)^n - 1 \] (converting the nominal rate to the annual effective rate)

REVISION EXERCISE

1. Craig deposits R65 000 into a savings account paying 15% per annum compounded monthly. He wants to buy a car in three years time.
   (a) Calculate how much money will be available to him in three years’ time, if he makes a further deposit of R10 000 into the account one year after his first deposit.
   (b) Craig then buys the car for the amount saved after three years. He drives the car for four years and then sells the car. Suppose that after four years of reducing balance depreciation, the car is worth one quarter of its original value. Calculate the depreciation interest rate as a percentage.

2. Calculate the original price of a laptop if its depreciated value after 7 years is R3200 and the rate of depreciation was 12% per annum based on the linear depreciation method.

3. Robyn invests R30 000 at 16% per annum compounded quarterly for a period of 15 years.
   (a) Convert the nominal rate of 16% per annum compounded quarterly to the equivalent effective rate (annual).
   (b) Now use the annual effective rate to calculate the future value of the investment after 15 years.

4. Pauline deposits R100 000 into a five year savings account. Two years later, she withdraws R20 000. Calculate the future value of her investment if the interest rate for the first three years is 18% per annum compounded monthly and 18% per annum compounded half-yearly for the remaining two years.

5. Sean repays a loan by means of two payments. The first repayment of R13 000 is made after four years. The second and final repayment amounting to R5 000 is made three years later. The interest rate during the first four years is 9% per annum compounded semi-annually. For the remaining three years, the interest rate changes to 12% per annum compounded monthly. How much did Sean originally borrow?
CALCULATING THE TIME PERIOD OF A LOAN OR INVESTMENT

EXAMPLE 1

(a) Matthew deposits R4 000 in a savings account paying 10% per annum compounded annually. How long will it take for the savings to double?

Solution

\[ A = P(1 + i)^n \]
\[ 8 \ 000 = 4 \ 000(1 + 0.10)^n \]
\[ \therefore 8 \ 000 = 4 \ 000(1.10)^n \]
\[ \therefore 2 = (1.10)^n \]
\[ \therefore (1.10)^n = 2 \]
\[ \therefore n = \log_{1.10} 2 \]
\[ \therefore n = 7.272540897 \]

The investment will take approximately 7 years and 3 months to double at an interest rate of 8% per annum compounded annually.

The 3 months was obtained by doing the following calculation:

\[ 0.272540897 \times 12 = 3 \text{ months} \]

(b) Suppose that the interest rate for Matthew’s investment was 10% per annum compounded half-yearly. How long will it take for the investment to double in this case?

Solution

Interest will now be calculated twice per year. This means that the quoted annual rate of 10% must be divided by 2.

The number of half-years in 1 year is 2.

The number of half-years in 2 years is \(2 \times 2\) half-years.

The number of half-years in 3 years is \(3 \times 2\) half-years.

Therefore, the number of half-years in \(n\) years is \(n \times 2\) half-years = \(2n\)

\[ A = P\left(1 + \frac{i}{2}\right)^{2n} \] (divide \(i\) by 2 and multiply \(n\) by 2)

\[ 8 \ 000 = 4 \ 000\left(1 + \frac{0.10}{2}\right)^{2n} \]
\[ \therefore 8 \ 000 = 4 \ 000(1.05)^{2n} \]
\[ \therefore 2 = (1.05)^{2n} \]
\[ \therefore (1.05)^{2n} = 2 \]
\[ \therefore 2n = \log_{1.05} 2 \]
\[ \therefore 2n = 14.20669908 \]
\[ \therefore n = 7.103349541 \]

The investment will take approximately 7 years and 1 month to double at an interest rate of 10% per annum compounded half-yearly.

The 1 month was obtained by doing the following calculation:

\[ 0.103349541 \times 12 \approx 1 \text{ month} \]

Alternatively, if we divide \(i\) by 2 and let the value of \(n\) represent the number of half-years, the calculation will be as follows:
\[ A = P \left( 1 + \frac{i}{2} \right)^n \] (divide \( i \) by 2 and let \( n \) equal the number of half-years)

\[ 8000 = 4000 \left( 1 + \frac{0.10}{2} \right)^n \]

\[ \therefore 8000 = 4000(1.05)^n \]

\[ \therefore 2 = (1.05)^n \]

\[ \therefore (1.05)^n = 2 \]

\[ \therefore n = \log_{1.05} 2 \]

\[ \therefore n = 14.20669908 \text{ half-years} \]

\[ \therefore n = \frac{14.20669908}{2} \text{ years} \]

\[ \therefore n = 7.103349541 \text{ years} \]

**EXAMPLE 2**

A motor car costing R200 000 depreciated at a rate of 8% per annum on the reducing balance method. Calculate how long it took for the car to depreciate to a value of R90 000 under these conditions.

**Solution**

\[ A = P(1 - i)^n \]

\[ \therefore 90000 = 200000(1 - 0.08)^n \]

\[ \therefore 90000 = 200000(0.92)^n \]

\[ \therefore 90000 = 200000(0.92)^n \]

\[ \therefore 0.45 = (0.92)^n \]

\[ \therefore n = \log_{0.92} 0.45 \]

\[ \therefore n = 9.576544593 \]

It will take approximately 9 years 7 months to depreciate to R90 000.

The way to calculate the months is as follows:

\[ 0.576544593 \times 12 = 6.918535116 \approx 7 \text{ months} \]

**EXERCISE 1**

1. Pauline deposits R10 000 in a savings account. Calculate how long it will take her to double her money if the interest rate is:
   (a) 7% per annum compounded annually.
   (b) 7% per annum compounded quarterly.
   (c) 7% per annum compounded half-yearly.
   (d) 7% per annum compounded monthly.
   (e) 7% per annum simple interest.
2. A small business had a number of computers which originally cost R70 000. The computers were then sold for R30 000. Calculate how long it took the computers to depreciate to R30 000 if the rate of depreciation was:
(a) 12% per annum reducing balance depreciation.
(b) 12% per annum linear depreciation.

3. Find the time taken for a certain sum of money to double if the interest rate is 12% per annum compounded semi-annually.

4. Simone opens an account at a local clothing store and she spends R5 000. The interest rate charged is 24% per annum compounded monthly. How long will it take her to owe R8 000, if she makes no prior repayments on the account?

5. Determine how many years it will take an investment of R2 000 to earn an amount of R1 920 in interest, if the investment was made at an interest rate of 13% per annum compounded monthly.

6. R3 000 depreciates at 9% per annum on the reducing balance scale to an amount of R1 872.10 over a period of time. What will the value of R3 000 be over the same time period if depreciation took place on a straight-line basis at 9% per annum?

7. Determine how many years it would take for the value of a car to depreciate to 25% of its original value, if the rate of depreciation, based on the reducing balance method, is 16% per annum.

8. A car cost R300 000 and depreciated over a period of five years to half its original amount.
(a) Calculate the annual rate of depreciation if it is based on the reducing balance method.
(b) How long will it then take for the car to depreciate once again to half its value? Assume that the depreciation rate is the same as in (a).

INVESTMENTS OR LOANS INVOLVING ANNUITIES

Definition: An annuity is a series of equal investment payments or loan repayments at regular intervals subject to a rate of interest over a period of time.

INVESTMENTS IN FUTURE VALUE ANNUITIES

In a future value annuity, money is invested at regular intervals in order to save money for the future. The magic of compound interest makes the investment grow in value, especially if the interest rate is above the current inflation rate. Typical future value annuities include retirement annuities, in which people save money each month so as to receive pension payments once they retire. Other future value annuities include five-year endowment policies, in which people save money over the short term.

EXAMPLE 3

Suppose that R1 000 is deposited into a bank account. One month later, a further R1 000 is deposited into the account and then a further R1 000 one month after this. If the interest rate is 6% per annum compounded monthly, how much will have been saved after two months?
There are a few different ways to do this calculation.

**Method 1**

In this method, the amount at $T_0$ will grow with interest to $T_1$.
An amount of R1 000 will then be added to the savings and the accumulated amount will then grow for a further month to $T_2$. A further R1 000 will be added to the savings. The accumulated amount (future value $F$) can then be calculated.

At $T_1$:
$$1000 \left(1 + \frac{0.06}{12}\right)^1 + 1000 = 2005$$

At $T_2$:
$$2005 \left(1 + \frac{0.06}{12}\right)^1 + 1000 = 3015.03$$

Another approach is:
$$F = \left[1000 \left(1 + \frac{0.06}{12}\right)^1 + 1000\right] \left(1 + \frac{0.06}{12}\right)^1 + 1000$$

$\therefore F = R3015.03$

**Method 2**

The amount at $T_0$ will grow for two months:

$$A = 1000 \left(1 + \frac{0.06}{12}\right)^2 = 1010.03$$

The amount at $T_1$ will grow for one month:

$$A = 1000 \left(1 + \frac{0.06}{12}\right)^1 = 1005$$

The amount at $T_2$ will have been deposited into the bank and no interest will have been earned for this payment as yet:

$$A = 1000 \left(1 + \frac{0.06}{12}\right)^0 = 1000$$

66
The total amount of money saved at the end of the second month will be the sum of the three payments. This amount is called the future value of the investment (F).

\[ F = 1000 \left(1 + \frac{0.06}{12}\right)^2 + 1000 \left(1 + \frac{0.06}{12}\right)^1 + 1000 \left(1 + \frac{0.06}{12}\right)^0 \]

\[ \therefore F = R3015.03 \]

**Method 3**

The series \[ F = 1000 \left(1 + \frac{0.06}{12}\right)^2 + 1000 \left(1 + \frac{0.06}{12}\right)^1 + 1000 \left(1 + \frac{0.06}{12}\right)^0 \] can be written as follows:

\[ F = 1000 \left(1 + \frac{0.06}{12}\right)^2 + 1000 \left(1 + \frac{0.06}{12}\right)^1 + 1000(1) \]

\[ \therefore F = 1000 + 1000 \left(1 + \frac{0.06}{12}\right)^1 + 1000 \left(1 + \frac{0.06}{12}\right)^2 \]

This is a geometric series with \( a = 1000 \) and \( r = \left(1 + \frac{0.06}{12}\right) \).

There is a total of three payments made in the two-month period \( (n = 3) \).

\[ S_n = \frac{a(r^n - 1)}{r - 1} \]

\[ S_3 = \frac{1000 \left[ \left(1 + \frac{0.06}{12}\right)^3 - 1 \right]}{\left(1 + \frac{0.06}{12}\right) - 1} = R3015.03 \]

**Method 4**

A formula referred to as the future value annuity formula can also be used to do the calculation. This formula is derived from the sum of the first \( n \) terms of a geometric series and is extremely useful in Financial Maths (will be derived later in the chapter).

\[ F = \frac{x(1 + i)^n - 1}{i} \]

where:

\( x = \) equal payments made per period

\( i = \) interest rate as a decimal \( = \frac{r}{100} \)

\( n = \) number of payments made

If we now refer to the previous example, the future value can be calculated as follows:
Notice that in this two-month savings period, a total of three payments were made. This is because a payment was made at $T_0$.

**Note:** If the duration of the savings is two months and payments are made at the end of each month (excluding a payment at $T_0$), then the number of payments made will be two rather than three.

The advantage of this formula is that we can calculate the future value when a whole lot of payments are made.

**EXAMPLE 4**

Suppose that R1 500 is invested every month, starting immediately, for a period of 12 months. Interest is 18% per annum compounded monthly. The future value of the investment after 12 months would be the sum of all payments together with the interest earned.

$$\frac{x [(1 + i)^n - 1]}{i} = \frac{1000 \left( \left(1 + \frac{0.06}{12} \right)^3 - 1 \right)}{0.06 \over 12}$$

$\therefore F = R3015.03$

Savings plan is two months.
The number of payments made is two.

The future value of this payment after 12 months (at $T_{12}$) can be calculated by using the formula $A = P(1 + i)^n$:

$A = 1500(1.015)^{12}$
Consider the payment of R1 500 at T_1:
The future value of this payment after 11 months (at T_{12}) can be calculated by using the formula A = P(1+i)^N:

\[ A = 1500(1,015)^{11} \]

Consider the payment of R1 500 at T_2:
The future value of this payment after 10 months (at T_{12}) can be calculated by using the formula A = P(1+i)^N:

\[ A = 1500(1,015)^{10} \]

We can continue to do this with all of the other payments, from T_3 to T_{11}.

Now consider the last payment of R1 500 at T_{12}:
This last payment will not earn any interest since the investment’s value is calculated as soon as this payment is made.
Here \( A = 1500(1,015)^0 \)

Each of the accumulated amounts can now be added to calculate what the future value of the investment will be at T_{12}.

\[ F = 1500(1,015)^{12} + 1500(1,015)^{11} + 1500(1,015)^{10} + \ldots + 1500(1,015)^1 + 1500(1,015)^0 \]

**Method 1**
The series is a geometric series and can be written as follows:
\[ F = 1500 + 1500(1,015)^1 + 1500(1,015)^2 + \ldots + 1500(1,015)^{11} + 1500(1,015)^{12} \]
where \( a = 1500 \) and \( r = 1,015 \)
By using the formula to calculate the sum of the first \( n \) terms of a geometric series, we can now easily calculate the value of this series. In this series there are 13 terms to be added. This is because a payment was made at T_0. The savings period is 12 months.
\[ F = a \left[ r^n - 1 \right] \]
\[ \frac{r - 1}{1} \]
\[ \therefore F = \frac{1500 \left[ (1,015)^{13} - 1 \right]}{1,015 - 1} = R21,355.24 \quad \left[ a = 1500, \quad r = 1,015 \right] \]

**Method 2**

A useful and highly recommended formula which helps us to add up these amounts quickly is the future value annuity formula.

\[
\begin{array}{cccccccccccccc}
T_0 & T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9 & T_{10} & T_{11} & T_{12} \\
1500 & 1500 & 1500 & 1500 & 1500 & 1500 & 1500 & 1500 & 1500 & 1500 & 1500 & 1500 & 1500 \\
\end{array}
\]

There are a total of 13 payments of R1 500 made in the 12 month savings period.

\[ F = \frac{x \left( (1+i)^n - 1 \right) \right]}{i} \]
\[ \therefore F = \frac{1500 \left[ (1,015)^{13} - 1 \right]}{0,015} = R21,355.24 \]

**Note:**

If the savings period is 12 months but the first payment is made at \( T_1 \) instead of \( T_0 \), the time line will then look as follows:

\[
\begin{array}{cccccccccccccc}
T_0 & T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9 & T_{10} & T_{11} & T_{12} \\
1500 & 1500 & 1500 & 1500 & 1500 & 1500 & 1500 & 1500 & 1500 & 1500 & 1500 & 1500 & 1500 \\
\end{array}
\]

In this case, there will only be 12 payments of R1500. The future value of the investment after the 12 months, starting from the time when the decision to save was made, i.e. \( T_0 \), can be calculated as follows:

\[ \therefore F = \frac{x \left( (1+i)^n - 1 \right) \right]}{i} \]
\[ \therefore F = \frac{1500 \left[ (1,015)^{12} - 1 \right]}{0,015} = R19,561.82 \]

**Derivation of the Future Value Annuity Formula**

**not for examination purposes**

\[
\begin{array}{cccccccccccccc}
T_0 & T_1 & T_2 & T_3 & T_4 & T_{n-1} & T_n \\
X & X & X & X & X & X & X \\
\end{array}
\]

\[ F = x + x(1+i) + x(1+i)^2 + x(1+i)^3 + \ldots + x(1+i)^{n-1} \]
\[ F = \frac{a(r^n - 1)}{r - 1} \quad \text{where} \quad r \neq 1 \]
\[ F = \frac{x(1+i)^n - 1}{i} \]

\[ F = \frac{x(1+i)^n - 1}{i} \]

\[ ∴ F = \frac{x(1+i)^n - 1}{i} \]

**EXAMPLE 5**

Thomas starts saving money in a Unit Trust fund. He immediately deposits R800 into the fund. Thereafter, at the end of each month, he deposits R800 into the fund and continues to do this for ten years. Interest is 8% per annum compounded monthly. Calculate the future value of his investment at the end of the ten-year period.

**Solution**

**Method 1** (Annuity formula)

There are a total of 121 payments of R800 during the ten-year (120 months) savings period (there is a payment at \( T_0 \)).

\[ F = \frac{x(1+i)^n - 1}{i} \]

\[ \begin{align*}
F &= \frac{800 \left(1 + \frac{0.08}{12}\right)^{121} - 1}{\frac{0.08}{12}} \\
&= R 148 \, 132.54
\end{align*} \]

**Method 2** (Geometric series)

The geometric series is:

\[ 800 + 800 \left(1 + \frac{0.08}{12}\right)^1 + 800 \left(1 + \frac{0.08}{12}\right)^2 + \ldots + 800 \left(1 + \frac{0.08}{12}\right)^{120} \]

with \( a = 800, \ r = \left(1 + \frac{0.08}{12}\right) \)

\[ F = \frac{a \left(r^n - 1\right)}{r - 1} \]

\[ \begin{align*}
F &= \frac{800 \left(1 + \frac{0.08}{12}\right)^{121} - 1}{\left(1 + \frac{0.08}{12}\right) - 1} \\
&= R 148 \, 132.54
\end{align*} \]
EXAMPLE 6

Patrick decided to start saving money for a period of eight years starting on 31st December 2009. At the end of January 2010 (in one month’s time), he deposited an amount of R2 300 into the savings plan. Thereafter, he continued making deposits of R2 300 at the end of each month for the planned eight year period. The interest rate remained fixed at 10% per annum compounded monthly. How much will he have saved at the end of his eight year plan which started on the 31st December 2009?

Solution

In this example, the duration of the loan is 8 years (96 months). However, the number of payments is 96 because of the first payment being made one month after the starting of the savings plan.

There are a total of 96 payments and the duration of the investment is 8 years (96 months).

Method 1  (Annuity formula)

The future value can now be calculated using the formula:

\[ F = \frac{x[(1+i)^n-1]}{i} \]

\[ F = \frac{2300\left[\left(1+\frac{0.10}{12}\right)^{96} - 1\right]}{\frac{0.10}{12}} = R336\ 216.47 \]

Method 2  (Geometric series)

The geometric series is:

\[ 2300 + 2300\left(1+\frac{0.10}{12}\right) + 2300\left(1+\frac{0.10}{12}\right)^2 + \ldots + 2300\left(1+\frac{0.10}{12}\right)^{95} \]

\[ F = \frac{a\left[r^n-1\right]}{r-1} \quad \left[a = 2300 \text{ and } r = \left(1+\frac{0.10}{12}\right)\right] \]

\[ F = \frac{2300\left[\left(1+\frac{0.10}{12}\right)^{96} - 1\right]}{\left(1+\frac{0.10}{12}\right)-1} = R336\ 216.47 \]
EXAMPLE 7

Jesse deposits R400 into an account paying 14% per annum compounded half-yearly. Six months later, she deposits R400 into the account. Six months after this, she deposits a further R400 into the account. She then continues to make half-yearly deposits of R400 into the account for a period of nine years from her first deposit of R400. Calculate the value of her savings at the end of the savings period.

Solution

6 months 19 payments of R400

\[
\begin{array}{c|c|c|c|c}
\text{T}_0 & \text{T}_1 & \text{T}_2 & \text{T}_3 & \text{T}_{18} \\
400 & 400 & 400 & 400 & 400 \\
\end{array}
\]

There are a total of 19 payments and the duration of the investment is 9 years (18 six-month periods).

Method 1 (Annuity formula)

The future value can now be calculated using the formula:

\[
F = \frac{400 \left(1 + \frac{0.14}{2}\right)^{19} - 1}{\frac{0.14}{2}} = R14\,951.59
\]

Method 2 (Geometric series)

The geometric series is:

\[
400 \left[400 \left(1 + \frac{0.14}{2}\right)^{19} - 1\right] = R14\,951.59
\]

EXAMPLE 8

Dayna has just turned 20 years old and has a dream of saving R8 000 000 by the time she reaches the age of 50. She starts to pay equal monthly amounts into a retirement annuity which pays 18% per annum compounded monthly. Her first payment starts on her 20th birthday and her last payment is made on her 50th birthday. How much will she pay each month?

Solution

\[
\begin{array}{c|c|c|c|c}
\text{T}_0 & \text{T}_1 & \text{T}_2 & \text{T}_3 & \text{T}_{360} \\
20th & 21st & 22nd & 23rd & 50th \\
\end{array}
\]

\[
\frac{0.18}{12} = 0.015
\]

\[
\text{S}_{358} \text{ S}_{359} \text{ S}_{360} = 8,000,000
\]
Method 1  (Annuity formula)
The savings is for a period of 30 years × 12 months = 360. The number of payments of \( x \) will be 361 (there is a payment at \( T_0 \)). The future value (F) in this example is R8 000 000.

\[
x = \frac{x[(1,015)^{361} - 1]}{0,015}
\]

\[
\therefore 8 \, 000 \, 000 = \frac{x[(1,015)^{361} - 1]}{0,015}
\]

\[
\therefore 8 \, 000 \, 000 \times 0,015 = x[(1,015)^{361} - 1]
\]

\[
\therefore \frac{8 \, 000 \, 000 \times 0,015}{(1,015)^{361} - 1} = x
\]

\[
\therefore x = R558,41
\]

Method 2  (Geometric series)
The geometric series is:

\[
x + x(1,015)^1 + x(1,015)^2 + \ldots + x(1,015)^{360}
\]

\[
F = \frac{a[r^n - 1]}{r - 1} \quad [a = x \text{ and } r = 1,015]
\]

\[
8000 \, 000 = \frac{x[(1,015)^{361} - 1]}{1,015 - 1}
\]

\[
\therefore 8000 \, 000 = \frac{x[(1,015)^{361} - 1]}{0,015}
\]

\[
\therefore \frac{8000 \, 000 \times 0,015}{(1,015)^{361} - 1} = x
\]

\[
\therefore x = R558,41
\]

EXERCISE 2

1. For each of the following time-lines, write down the number of payments.

(a) \[ T_0 \quad T_1 \quad T_2 \quad T_3 \]

(b) \[ T_0 \quad T_1 \quad T_2 \quad T_3 \]

(c) \[ T_0 \quad T_1 \quad T_2 \quad T_3 \quad T_4 \quad T_5 \quad T_6 \quad T_7 \quad T_8 \]

(d) \[ T_0 \quad T_1 \quad T_2 \quad T_3 \quad T_4 \quad T_5 \quad T_6 \quad T_7 \quad T_8 \]

(e) \[ T_0 \quad T_1 \quad T_2 \quad T_3 \quad T_4 \quad T_5 \quad T_6 \quad T_7 \quad T_8 \]
2. John deposits R500 into a savings account. At the end of each month thereafter, he deposits R500 into the account and continues to do this for six years. Interest is 7% per annum compounded monthly. Calculate the future value of his investment at the end of the six-year period.

3. Nandile deposits R1 800 into a savings account. At the end of each month thereafter, she deposits R1 800 into the account and continues to do this for ten years. Interest is 8% per annum compounded monthly. Calculate the future value of her investment at the end of the ten-year period.

4. Michael deposits R10 000 into a savings account. At the end of each year thereafter, he deposits R10 000 into the account and continues to do this for twenty years. Interest is 18% per annum effective. Calculate the future value of his investment at the end of the twenty-year period.

5. Pauline decided to start saving money for a period of twelve years starting on 31st December 2011. At the end of January 2012 (in one month’s time), she deposited R2 500 into the savings account. Thereafter, she continued
making deposits of R2 500 at the end of each month for the planned twelve-year period. The interest rate remained fixed at 10% per annum compounded monthly. How much will she have saved at the end of her twelve-year plan which started on the 31st December 2011?

6. Nerina deposits R1200 into an account paying 12% per annum compounded half-yearly. Six months later, she deposits R1 200 into the account. Six months after this, she deposits a further R1 200 into the account. She then continues to make half-yearly deposits of R1 200 into the account for a period of seven years from her first deposit of R1 200. Calculate the value of her savings at the end of the savings period.

7. Lawrence decides to start saving for a car. On his 15th birthday, he deposits R6 000 into a bank account with an interest rate of 8% per annum compounded quarterly. He continues to make quarterly payments until the last payment on his 24th birthday. How much money will he have saved by then to finance the purchase of a new car?

8. Athaliah wants to save up R400 000 in five years’ time in order to purchase a new car. She starts making monthly payments into an account paying 12% per annum compounded monthly, starting immediately. How much will she pay each month?

9. Sibongile decides to invest money into the share market in order to become a millionaire in ten years time. She manages to secure an interest rate of 24% per annum compounded monthly. In one month’s time, she starts making monthly payments into a share market account. How much must she invest per month in order to obtain R1 000 000 at the end of the twenty year savings plan?

10. Paul pays R3 000 into an eight-year savings account at the end of each month starting three months after his decision to start saving. The interest is 18% per annum compounded monthly. If the investment period, starting from his decision to start saving, is eight years, calculate the future value of the investment at the end of the eighth year.

**EXAMPLE 9**

A father decides to start a savings plan for his baby daughter’s future education. On opening the account, he immediately deposits R2 000 into the account and continues to make monthly payments at the end of each month thereafter for a period of 16 years. The interest rate remains fixed at 15% per annum compounded monthly.

(a) How much money will he have accumulated at the end of the 16th year?

(b) At the end of the sixteen-year period, he leaves the money in the account for a further year. How much money will he then have accumulated?

**Solutions**

\[
\begin{align*}
F = & \quad 2000 \quad 2000 \quad 2000 \quad 2000 \\
& \quad T_0 \quad T_1 \quad T_2 \quad T_3 \\
& \frac{0.15}{12} = 0.0125
\end{align*}
\]

There are 192 months in a period of 16 years (16 years × 12 months = 192 months). The number of payments made by the father is 193.
Method 1  (Annuity formula)

\[ F = \frac{2000\left[(1,0125)^{193} - 1\right]}{0,0125} = R1\ 599\ 386,02 \]

Method 2  (Geometric series)

The geometric series is:
\[ 2000 + 2000(1,0125)^1 + 2000(1,0125)^2 + \ldots \ldots + 2000(1,0125)^{192} \]

\[ F = \frac{a\left[r^n - 1\right]}{r - 1} \quad [a = 2000 \text{ and } r = 1,0125] \]

\[ \therefore F = \frac{2000\left[(1,0125)^{193} - 1\right]}{1,0125 - 1} = R1\ 599\ 386,02 \]

(b) Since there will no longer be any further payments of R2000 into the annuity, all we now need to do is grow the R1 599 386,02 for 12 months using the formula \( A = P(1+i)^n \) to calculate the future value of the investment after the further 12 months.

Annuity payments made in advance

It is often the case that in annuity investments, payments are made in advance. This means that the last payment in the annuity is made one month before the investment is paid out. The next example deals with this type of annuity.

EXAMPLE 10

In order to supplement his state pension after retirement, a school teacher aged 30 takes out a retirement annuity. He makes monthly payments of R1 000 into the fund and the payments start immediately. The payments are made in advance, which means that the last payment of R1 000 is made one month before the annuity pays out. The interest rate for the annuity is 12% per annum compounded monthly. Calculate the future value of the annuity in twenty-five years’ time.
A total of 300 payments will be made.

**Method 1  (Annuity formula)**

The value of the annuity will first be calculated at $T_{299}$. The accumulated amount at $T_{299}$ will then be grown for one month.

$$F = \frac{1000[(1.01)^{300} - 1]}{0.01}$$

$$\therefore F = 1\,878\,846.626$$

$$A = 1\,878\,846.626(1.01)^{1} = R1\,897\,635.09$$

A more efficient approach:

$$A = \frac{1000[(1.01)^{300} - 1]}{0.01} \cdot (1.01)^{1}$$

$$\therefore A = R1\,897\,635.09$$

**Method 2  (Geometric series)**

The geometric series is: $1000(1.01)^{1} + 1000(1.01)^{2} + \ldots + 1000(1.01)^{300}$

$$F = \frac{a(r^{n} - 1)}{r - 1} \quad [a = 1000(1.01) \text{ and } r = 1.015]$$

$$\therefore F = \frac{1000(1.01)[(1.015)^{300} - 1]}{1.015 - 1} = R1\,897\,635.09$$

**EXAMPLE 11  (Enrichment)**

It is the 31st December 2010. Anna decides to start saving money and wants to save R300 000 by paying monthly amounts of R4000, starting in one month’s time (on 31st January 2011), into a savings account paying 15% per annum compounded monthly. How many payments of R4000 will be made? The duration of the savings starts on the 31st December 2010, even though the first payment is not made on the 31st December 2010.

**Solution**

$$\frac{0.15}{12} = 0.0125$$
There are \( n \) number of payments of R4000.

**Method 1**  (Annuity formula)

\[
300 \, 000 = \frac{4000 \left[ (1,0125)^n - 1 \right]}{0,0125}
\]

\[
\therefore \frac{300 \, 000 \times 0,0125}{4000} = (1,0125)^n - 1
\]

\[
\therefore \frac{300 \, 000 \times 0,0125}{4000} + 1 = (1,0125)^n
\]

\[
\therefore 1,9375 = (1,0125)^n
\]

\[
\therefore (1,0125)^n = 1,9375
\]

\[
\therefore n = \log_{1,0125} 1,9375
\]

\[
\therefore n = 53,24189314
\]

**Method 2**  (Geometric series)

The geometric series is:

\[
4000 + 4000(1,0125)^1 + \ldots + 4000(1,0125)^{n-1}
\]

\[
300 \, 000 = \frac{4000 \left[ (1,0125)^n - 1 \right]}{1,0125 - 1}
\]

\[
\therefore \frac{300 \, 000 \times 0,0125}{4000} = (1,0125)^n - 1
\]

\[
\therefore \frac{300 \, 000 \times 0,0125}{4000} + 1 = (1,0125)^n
\]

\[
\therefore 1,9375 = (1,0125)^n
\]

\[
\therefore (1,0125)^n = 1,9375
\]

\[
\therefore n = \log_{1,0125} 1,9375
\]

\[
\therefore n = 53,24189314
\]

The value of \( n \) represents the number of payments made. This means that Anna will make 53 payments of R4 000, but what does the decimal represent?

Let’s explore this on a time-line.

![Time-line diagram](image)

The amount accumulated after 53 months is:

\[
\frac{4000 \left[ (1,0125)^{53} - 1 \right]}{0,0125} = 298139,7445
\]

This amount is clearly less than the R300 000 required.

The amount accumulated after 54 months is:

\[
\frac{4000 \left[ (1,0125)^{53} - 1 \right]}{0,0125} (1,0125)^1 = 301866,4913
\]

This amount is clearly more than the R300 000 required.

Therefore, if 53 payments of R4 000 are made and the accumulated amount is left to grow for a few days (or weeks) into the next month, the amount of R300 000 will be acquired. At the end of the 54th month, the investor will have saved more than R300 000. Therefore there will be 53 payments of R4 000 and the accumulated amount will need to grow into the next month in order for the R300 000 to be obtained.
EXERCISE 3

1. Simon starts saving money in a Unit Trust fund. He immediately deposits R900 into the fund. Thereafter, at the end of each month, he deposits R900 into the fund and continues to do this for the five years. Interest is 18% per annum compounded monthly.
   (a) Calculate the future value of his investment at the end of the five-year period.
   (b) Simon leaves his money in the fund to grow for two years without making any further payments of R900. The interest rate changes to 10% per annum compounded quarterly. Calculate the value of his investment after the two-year period.

2. Irene deposits R1 700 into a savings account. At the end of each month thereafter, she deposits R1 700 into the account and continues to do this for ten years. Interest is 8% per annum compounded monthly.
   (a) Calculate the future value of her investment at the end of the ten-year period.
   (b) Irene leaves the accumulated amount in the account for a further six months without making any further payments of R1 700. If the interest rate changes to 10% per annum compounded half-yearly, calculate the value of the investment after the further six months.

3. In order to supplement his pension after retirement, Mpho (aged 20) takes out a retirement annuity. He makes monthly payments of R2 000 into the fund and the payments start immediately. The payments are made in advance, which means that the last payment of R2 000 is made one month before the annuity pays out. The interest rate for the annuity is 15% per annum compounded monthly. Calculate the future value of the annuity when he turns 60.

4. Katlego starts a five year savings plan. At the beginning of the month he deposits R2 000 into the account and makes a further deposit of R2 000 at the end of that month. He then continues to make month end payments of R2 000 into the account for the five year period (starting from his first deposit). The interest rate is 6% per annum compounded monthly.
   (a) Calculate the future value of his investment at the end of the five year period.
   (b) Due to financial difficulty, Katlego misses the last two payments of R2000. What will the value of his investment now be at the end of the five year period?

5. R500 is invested each month, starting in one month’s time, into an account paying 16% per annum compounded monthly. The fund accumulates to R 10 000. How many payments of R500 will be made?

6. R1 000 is invested every three months, starting in three month’s time, into an account paying 14% per annum compounded quarterly. The fund accumulates to R25 000. How many payments of R1 000 will be made?

7. R2 000 is immediately deposited into a savings account. Six months later and every six months thereafter, R2 000 is deposited into the account. The interest rate is 16% per annum compounded half-yearly. The future value of the savings is R100 000. How many payments of R2 000 will be made?
A DISCUSSION OF LOANS IN THE WORLD OF FINANCE

HIRE PURCHASE LOANS

In Grade 10 and 11, you studied hire purchase loans in which the interest payable is calculated at the start of the loan using the formula for simple interest. The total interest on the loan must be paid in full. This means that if you want to pay off the loan earlier than the loan period, you will still need to pay the full amount of interest owed. The monthly payments are calculated by dividing the loan amount by the number of months. You are required to pay all of these amounts and there is no advantage in paying off the loan early.

EXAMPLE 12

Sally buys a tumble dryer for R4 000. She takes out a hire-purchase loan involving equal monthly payments over three years. The interest rate charged is 14% per annum simple interest. Calculate:
(a) the actual amount paid for the tumble dryer.
(b) the interest paid.
(c) how much must be paid each month.

Solutions

(a) \[ A = P(1 + in) \]
\[ A = 4000\left[1 + (0,14)(3)\right] = R5680 \]

(b) Interest paid is \[ R5680 - R4000 = R1680 \]

(c) The monthly loan repayments:
\[ \frac{5680}{36} = R157,78 \]

The total amount of R5 680 must be paid even if Sally wants to pay off the loan early.

REDUCING BALANCE LOANS

In bank loans, interest is paid on the reducing balance. The lower the balance, the less interest you have to pay. The advantage of these loans is that any additional payments into the loan will reduce the amount owed. By paying additional payments into the loan, you are able to reduce the duration of the loan as well as save a lot of interest. The next example will help to explain how this loan works.

EXAMPLE 13

James takes out a one year bank loan of R18 000 to pay for an expensive laptop. The interest rate is 18% per annum compounded monthly and monthly repayments of R1 650,24 are made starting one month after the granting of the loan.
We will now demonstrate how this reducing balance loan is paid off and the advantage of taking out this loan rather than a hire purchase loan.

<table>
<thead>
<tr>
<th>End of the first month</th>
<th>18000 + 0,015×18000 – 1650,24 = 16619,76</th>
</tr>
</thead>
<tbody>
<tr>
<td>End of the second month</td>
<td>16619,76 + 0,015×16619,76 – 1650,24 = 15218,8164</td>
</tr>
<tr>
<td>End of the third month</td>
<td>15218,8164 + 0,015×15218,8164 – 1650,24 = 13796,85865</td>
</tr>
<tr>
<td>End of the fourth month</td>
<td>$13796.85865 + 0.015 \times 13796.85865 - 1650.24 = 12353.57153$</td>
</tr>
<tr>
<td>End of the fifth month</td>
<td>$12353.57153 + 0.015 \times 12353.57153 - 1650.24 = 10888.6351$</td>
</tr>
<tr>
<td>End of the sixth month</td>
<td>$10888.6351 + 0.015 \times 10888.6351 - 1650.24 = 9401.72425$</td>
</tr>
<tr>
<td>End of the seventh month</td>
<td>$9401.72425 + 0.015 \times 9401.72425 - 1650.24 = 7892.510494$</td>
</tr>
<tr>
<td>End of the eighth month</td>
<td>$7892.510494 + 0.015 \times 7892.510494 - 1650.24 = 6360.658152$</td>
</tr>
<tr>
<td>End of the ninth month</td>
<td>$6360.658152 + 0.015 \times 6360.658152 - 1650.24 = 4805.828024$</td>
</tr>
<tr>
<td>End of the tenth month</td>
<td>$4805.828024 + 0.015 \times 4805.828024 - 1650.24 = 3227.675444$</td>
</tr>
<tr>
<td>End of the eleventh month</td>
<td>$3227.675444 + 0.015 \times 3227.675444 - 1650.24 = 1625.850576$</td>
</tr>
<tr>
<td>End of the twelfth month</td>
<td>$1625.850576 + 0.015 \times 1625.850576 - 1650.24 = 0$</td>
</tr>
</tbody>
</table>

**Note:**

(a) The interest at the end of each month is calculated on the previous balance, which gets less and less each time.

(b) If an additional amount is paid into the loan at any stage, this will mean that the balance will be less and therefore the interest paid will be calculated on a lesser amount. In this way, the loan can be paid off quickly saving you a lot of money.

(c) The total amount of interest paid on this loan is calculated as follows:

Total interest paid = Total amount paid − original amount borrowed

$\therefore$ Total interest paid = $1650.24 \times 12 − 18000

$\therefore$ Total interest paid = $19802.88 − 18000

$\therefore$ Total interest paid = R1802.88

**PRESENT VALUE ANNUITIES**

A reducing balance loan is often referred to as a present value annuity. In a present value annuity, a sum of money is normally borrowed from a financial institution and paid back with interest by means of regular payments at equal intervals over a time period. The loan is said to be amortised (paid off) when it together with interest charges is paid off. The interest is calculated on the reducing balance. We will now develop the present value annuity formula which is used in calculations involving reducing balance loans.

**EXAMPLE 14**

Suppose that a loan is repaid by means of a payment of R1 000 one month after the loan was granted and one further payment of R1 000 one month after the first payment of R1 000. Calculate the amount borrowed (the present value of the loan). The interest rate is 6% per annum compounded monthly.
Method 1

The present value at $T_0$ of the payment at $T_1$ can be calculated as follows:

$$P = 1000 \left(1 + \frac{0.06}{12}\right)^{-1} = 995.02$$

The present value at $T_0$ of the payment at $T_2$ can be calculated as follows:

$$P = 1000 \left(1 + \frac{0.06}{12}\right)^{-2} = 990.0745031$$

The amount borrowed is called the loan ($P$) is the sum of the two present values:

$$P = 1000 \left(1 + \frac{0.06}{12}\right)^{-1} + 1000 \left(1 + \frac{0.06}{12}\right)^{-2} = \text{R1 985,10}$$

You will probably easily notice that this series is a geometric series.

By using the formula to calculate the sum of the first $n$ terms of a geometric series, we can now easily calculate the value of this series. In this series there are two terms to be added.

$$P = \frac{a \left(r^n - 1\right)}{r - 1}$$

$$\therefore P = \frac{1000 \left(1 + \frac{0.06}{12}\right)^{-1} \left[\left(1 + \frac{0.06}{12}\right)^{-2} - 1\right]}{\left(1 + \frac{0.06}{12}\right)^{-1} - 1}$$

$$\therefore P = \text{R1 985,10}$$

Method 2

A formula referred to as the present value annuity formula can also be used to do the calculation. This formula is derived from the sum of the first $n$ terms of a geometric series and is extremely useful in Financial Maths (will be derived later in the chapter).

$$P = \frac{x \left[1 - (1 + i)^{-n}\right]}{i}$$

where:

- $x =$ equal payments made per period
- $i =$ interest rate as a decimal $= \frac{r}{100}$
- $n =$ number of payments made

If we now refer to the previous example, the present value can be calculated as follows:

$$P = \frac{x \left[1 - (1 + i)^{-n}\right]}{i}$$

$$1000 \left[1 - \left(1 + \frac{0.06}{12}\right)^{-2}\right]$$

$$\therefore P = \frac{0.06}{12}$$

$$\therefore P = \text{R1 985,10}$$
The advantage of this formula is that we can calculate the present value when a whole lot of payments are made.

**EXAMPLE 15**

James takes out a one year bank loan to pay for an expensive laptop. The interest rate is 18% per annum compounded monthly and monthly repayments of R1 650,24 are made starting one month after the granting of the loan. Show that he borrowed an amount of R18 000.

\[
P = 1650.24(1.015) - 1650.24(1.015) - 1650.24(1.015) - 1650.24(1.015)
\]

Consider the payment of R1 650,24 at \(T_1\):
The present value of this payment at \(T_0\) can be calculated by using the formula

\[
P = A(1 + i)^{-n}:
P = 1650.24(1.015)^{-1}
\]

Consider the payment of R1 650,24 at \(T_2\):
The present value of this payment at \(T_0\) can be calculated by using the formula

\[
P = A(1 + i)^{-n}:
P = 1650.24(1.015)^{-2}
\]

Consider the payment of R1 650,24 at \(T_3\):
The present value of this payment at \(T_0\) can be calculated by using the formula

\[
P = A(1 + i)^{-n}:
P = 1650.24(1.015)^{-3}
\]

We can continue to do this with all of the other payments, from \(T_4\) to \(T_{12}\).

Each of the calculated present value amounts can now be added to calculate what the present value of the loan will be at \(T_0\).

\[
P = 1650.24(1.015)^{-1} + 1650.24(1.015)^{-2} + 1650.24(1.015)^{-3} + \ldots + 1650.24(1.015)^{-12}
\]
Method 1
You will probably easily notice that this series is a geometric series.
By using the formula to calculate the sum of the first $n$ terms of a geometric series, we can now easily calculate the value of this series. In this series there are 12 terms to be added.

\[
P = \frac{a\left(r^n - 1\right)}{r - 1}
\]

\[
\therefore P = \frac{1650.24(1.015)^{-1}\left[(1.015)^{-12} - 1\right]}{(1.015)^{-1} - 1} \quad [a = 1650.24(1.015)^{-1}, \quad r = (1.015)^{-1}]
\]

\[
\therefore P = \text{R18 000}
\]

Method 2
A useful and recommended formula to calculate the sum of the present values of these payments at $T_0$ is the present value annuity formula:

\[
P = \frac{1650.24\left[1 - (1.015)^{-12}\right]}{0.015} = \text{R18 000}
\]

Derivation of the Present Value Annuity Formula
(not for examination purposes)

\[
P = x(1+i)^{-1} + x(1+i)^{-2} + x(1+i)^{-3} + \ldots + x(1+i)^{-n}
\]

\[
P = \frac{a\left(r^n - 1\right)}{r - 1}
\]

\[
\therefore P = \frac{x(1+i)^{-1}\left[(1+i)^{-1}\right]^n - 1}{(1+i)^{-1} - 1} \quad [a = x(1+i)^{-1}, \quad r = (1+i)^{-1}]
\]

\[
\therefore P = \frac{x\left[(1+i)^{-n} - 1\right]}{(1+i)^{1} - (1+i)^{-1}}
\]

\[
\therefore P = \frac{x\left[(1+i)^{-n} - 1\right]}{(1+i)^{0} - (1+i)^{1}}
\]

\[
\therefore P = \frac{x\left[(1+i)^{-n} - 1\right]}{1 - (1+i)}
\]

\[
\therefore P = \frac{x\left[(1+i)^{-n} - 1\right]}{-i}
\]

\[
\therefore P = \frac{x\left[1 - (1+i)^{-n}\right]}{i}
\]
Note:
The formula for P can only be used if there is a gap between the loan and the first payment. If the payments are monthly, then there must be a one month gap between the loan and the first payment.

EXAMPLE 16
Malibongwe takes out a bank loan to pay for his new car. He repays the loan by means of monthly payments of R5 000 for a period of five years starting one month after the granting of the loan. The interest rate is 24% per annum compounded monthly. Calculate the purchase price of his new car.

Solution

\[
\begin{array}{cccccccc}
\text{P} & \text{gap} & 5000 & 5000 & 5000 & 5000 & 5000 \\
T_0 & T_1 & T_2 & T_3 & T_{58} & T_{59} & T_{60} \\
\end{array}
\]

\[
\frac{0.24}{12} = 0.02
\]

There are 60 months in five years and the number of payments are 60.

Method 1 (Annuity formula)

\[
P = \frac{5000\left[1-(1,02)^{-60}\right]}{0.02}
\]

\[
\therefore P = \text{R173 804,43}
\]

Method 2 (Geometric series)

The geometric series is:

\[
5000(1,02)^{-1} + 5000(1,02)^{-2} + \ldots + 5000(1,02)^{-60}
\]

\[
P = \frac{5000(1,02)^{-1}\left[1-(1,02)^{-60}\right]}{(1,02)^{-1}-1}
\]

\[
\left[a = 5000(1,02)^{-1}, \quad r = (1,02)^{-1}\right]
\]

\[
\therefore P = \text{R173 804,43}
\]

Note:
If a loan is taken out and a payment is made at the same time, then this payment must be subtracted from the original loan. This payment is really a deposit and must be deducted from the loan before applying the present value annuity formula.

EXAMPLE 17
Malibongwe takes out a bank loan to pay for his new car. He pays an initial amount (deposit) of R10 000. He then makes monthly payments for a period of five years starting one month after the granting of the loan. The interest rate is 24% per annum compounded monthly. Calculate the monthly payments if the car originally cost him R173 804,43.

Solution

\[
173 804,43 - 10 000
\]

\[
\frac{0.24}{12} = 0.02
\]
Method 1  (Annuity formula)

\[ 173\,804.43 - 10\,000 = \frac{x\left[1 - (1.02)^{-60}\right]}{0.02} \]

\[ \therefore \quad 163\,804.43 \times 0.02 \left[1 - (1.02)^{-60}\right] = x \]

\[ \therefore \quad x = R4\,712.32 \]

Method 2  (Geometric series)

The geometric series is: \[x(1.02)^{-1} + x(1.02)^{-2} + \ldots + x(1.02)^{-60}\]

\[ 173\,804.43 - 10\,000 = \frac{x(1.02)^{-1}\left[(1.02)^{-60} - 1\right]}{(1.02)^{-1} - 1} \]

\[ \therefore \quad \frac{163\,804.43 \times (1.02)^{-1} - 1}{(1.02)^{-1} - 1} = x \]

\[ \therefore \quad x = R4\,712.32 \]

Another type of present value annuity is one in which you invest a large sum of money into a bank and the bank pays you monthly payments with interest. In these annuities, you lend money to the bank and the bank pays you back with interest. When you borrow money from the bank, you pay the bank monthly payments with interest. In both situations, the large sum of money is the present value and therefore you have to use the present value annuity formula to do calculations with these amounts. When you are saving to receive a large sum of money in the future, then you are working with the future value annuity formula.

EXAMPLE 18

How much money will you need to win in the National Lottery so as to receive equal monthly payments of R10 000 per month from a bank for a period of twenty years starting one month after winning the money? The bank grants you an interest rate of 12% per annum compounded monthly.

Solution

This is clearly a present value annuity, because you have a large sum of money to loan to the bank. You will now receive monthly payments with interest rather than you paying the bank for money that you might have borrowed now. This is a great investment. Let’s hope that you can win the Lotto soon!

Method 1  (Annuity formula)

\[ P = \frac{10\,000\left[1 - (1.01)^{-240}\right]}{0.01} \]

\[ \therefore \quad P = R908\,194.16 \]
Method 2  (Geometric series)

The geometric series is: \[10000(1,01)^{-1} + 10000(1,01)^{-2} + \ldots + 10000(1,01)^{-240}\]

\[
P = \frac{10000(1,01)^{-1} \left[ (1,01)^{-240} - 1 \right]}{(1,01)^{-1} - 1}
\]

\[a = 10000(1,01)^{-1}, \quad r = (1,01)^{-1}\]

\[\therefore P = R908\ 194,16\]

Summary of the annuity formulae

<table>
<thead>
<tr>
<th>Future value annuity formula</th>
<th>Present value annuity formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>[F = \frac{x[(1+i)^n - 1]}{i}]</td>
<td>[P = \frac{x[1-(1+i)^{-n}]}{i}]</td>
</tr>
<tr>
<td>where: [x = \text{payment per period}]</td>
<td>where: [x = \text{payment per period}]</td>
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<tr>
<td>[n = \text{number of payments}]</td>
<td>[n = \text{number of payments}]</td>
</tr>
<tr>
<td>[i = \text{interest rate}]</td>
<td>[i = \text{interest rate}]</td>
</tr>
</tbody>
</table>

Note:
The formula for \(P\) can only be used if there is a gap between the loan and the first payment. The formula for \(F\) can only be used if there is a final payment at the end, which doesn’t earn interest. \(F\) coincides with the \(x\) at \(T_n\).

EXERCISE 4

1. David takes out a bank loan in order to pay for his new car. He repays the loan by means of monthly payments of R5 500 for a period of five years starting one month after the granting of the loan. The interest rate is 21% per annum compounded monthly. Calculate the purchase price of his new car.

2. How much can Brenda borrow from a bank if she repays the loan by means of equal monthly payments of R4 500, starting in one month’s time? The interest rate is 12% per annum compounded monthly and the duration of the loan is eight years.

3. How much can Lerato borrow from a bank if she repays the loan by means of equal quarterly payments of R2 000, starting in three months time? The interest rate is 18% per annum compounded quarterly and the duration of the loan is ten years.

4. Brenda takes out a twenty year loan of R400 000. She repays the loan by means of equal monthly payments starting one month after the granting of the loan. The interest rate is 16% per annum compounded monthly. Calculate the monthly repayments.
5. Lerato plans to buy a car for R125 000. She pays a deposit of 15% and takes out a bank loan for the balance. The bank charges 12.5% per annum compounded monthly. Calculate the monthly repayment on the car if the loan is repaid over the six-year period.

6. Simphiwe takes out a ten year loan of R100 000. She repays the loan by means of equal monthly payments starting one month after the granting of the loan. The interest rate is 18% per annum compounded monthly. Calculate the monthly payments.

7. Kevin takes out a bank loan to pay for his new car. He repays the loan by means of monthly payments of R3 000 for a period of six years starting one month after the granting of the loan. The interest rate is 18% per annum compounded monthly.
   (a) Calculate the purchase price of his new car.
   (b) If he paid an initial deposit of R2 000, what will his reduced monthly payments be?

8. Twenty-five half-yearly payments are made, starting six months time, in order to repay a loan of R100 000. What is the value of each payment if interest is 14% per annum compounded half-yearly?

9. In order for Sean to receive equal payments of R2 000 per month from the bank for a period of three years, starting in one month’s time, what amount will he need to invest now? Interest is 18% per annum compounded monthly.

10. Mark inherits R1 000 000 from his late uncle. He invests the money at an interest rate of 14% per annum compounded monthly. He wishes to earn a monthly salary from the investment for a period of twenty years starting in one month’s time. How much will he receive each month?

11. Brian takes out a retirement annuity that will supplement his pension when he retires in thirty years’ time. He estimates that he will need R2,5 million in this retirement fund at that stage. The interest rate he earns is 9% per annum compounded monthly.
   (a) Calculate his monthly payment into this fund if he starts paying immediately and makes his final payment in 30 years’ time.
   (b) The retirement fund does not pay out the R2,5 million when Brian retires. Instead he will be paid monthly amounts, for a period of twenty years, starting one month after his retirement. If the interest that he earns over this period is calculated at 7% per annum compounded monthly, determine the monthly payments he will receive.

EXAMPLE 19

Melanie takes out a twenty year loan of R100 000. She repays the loan by means of equal monthly payments starting three months after the granting of the loan. The interest rate is 18% per annum compounded monthly. Calculate the monthly payments.

**Solution**
Method 1  (Annuity formula)

The present value formula only works if there is a gap between the loan and the first payment. Therefore, it is necessary to first grow the loan to \( T_2 \), which is a gap before the first payment. The number of payments in this deferred annuity will therefore only be 238, because two are missing (at \( T_1 \) and \( T_2 \)).

\[
100 000(1,015)^2 = \frac{x\left[1 - (1,015)^{-238}\right]}{0,015}
\]

\[
\therefore 100 000(1,015)^2 \times 0,015 = x\left[1 - (1,015)^{-238}\right]
\]

\[
\therefore \frac{100 000(1,015)^2 \times 0,015}{1 - (1,015)^{-238}} = x
\]

\[
\therefore x = R1591,35
\]

Method 2  (Geometric series)

The geometric series is:

\[
\frac{340}{(1,015)} + \frac{340}{(1,015)^2} + \frac{340}{(1,015)^3} + \cdots
\]

\[
3238 \left(\frac{1}{1,015}\right) \left(\frac{1}{1,015}\right) \cdots
\]

\[
\left(\frac{1}{1,015}\right) \left(\frac{1}{1,015}\right) \left(\frac{1}{1,015}\right)
\]

\[
\frac{100 000}{(1,015)^{-3} \left(1,015^2 - 1\right)} = x
\]

\[
\therefore x = R1591,35
\]

EXAMPLE 20  (Enrichment)

Peter borrows R500 000 from a bank and repays the loan by means of monthly payments of R8 000, starting one month after the granting of the loan. Interest is fixed at 18% per annum compounded monthly.

(a) How many payments of R8 000 will be made and what will the final lesser payment be?

(b) How long is the savings period?

Solutions

(a) 

\[
\begin{align*}
\text{Method 1} & \quad \text{(Annuity formula)} \\
500 000 & = \frac{8000\left[1 - (1,015)^{-n}\right]}{0,015} \\
\therefore \frac{500 000 \times 0,015}{8000} & = 1 - (1,015)^{-n} \\
\therefore (1,015)^{-n} & = 1 - \frac{500 000 \times 0,015}{8000}
\end{align*}
\]

\[
\text{Method 2} \quad \text{(Geometric series)}
\]

\[
8000(1,015)^{-1} + \ldots + 8000(1,015)^{-n}
\]

\[
\therefore 500 000 = \frac{8000(1,015)^{-1}\left[(1,015)^{-n} - 1\right]}{(1,015)^{-1} - 1}
\]

\[
\therefore \frac{500 000 \left[(1,015)^{-1} - 1\right]}{8000(1,015)^{-1}} = (1,015)^{-n} - 1
\]
\[ \therefore (1,015)^{-n} = 0,0625 \]
\[ \therefore -n = \log_{1,015}0,0625 \]
\[ \therefore -n = 186,2221025 \]
\[ \therefore n = 186,2221025 \]

There will be 186 payments of R8 000 into the annuity. The decimal here indicates that there will be a final payment which is less than R8 000. The final payment, call it \( x \), can be calculated as follows:

**Method 1** (Annuity formula)

\[
500\,000 = \frac{8000\left[1 - (1,015)^{-186}\right]}{0,015} + x(1,015)^{-187}
\]
\[ \therefore 500\,000 - \frac{8000\left[1 - (1,015)^{-186}\right]}{0,015} = x(1,015)^{-187} \]
\[ \therefore 110,4090631 = x(1,015)^{-187} \]
\[ \therefore \frac{110,4090631}{(1,015)^{-187}} = x \]
\[ \therefore x = R1787,12 \]

**Method 2** (Geometric series)

The geometric series is: \( x(1,015)^{-3} + x(1,015)^{-4} + \ldots + x(1,015)^{-240} \)

\[
500\,000 = \frac{8000(1,015)^{-1}\left[(1,015)^{-186} - 1\right]}{(1,015)^{-1} - 1} + x(1,015)^{-187}
\]
\[ \therefore 500\,000 - \frac{8000(1,015)^{-1}\left[(1,015)^{-186} - 1\right]}{(1,015)^{-1} - 1} = x(1,015)^{-187} \]
\[ \therefore 110,4090631 = x(1,015)^{-187} \]
\[ \therefore \frac{110,4090631}{(1,015)^{-187}} = x \]
\[ \therefore x = R1787,12 \]

(b) The loan will therefore take 187 months to pay off.
EXERCISE 5

1. Mandy takes out a ten-year loan of R80 000. She repays the loan by means of equal monthly payments starting four months after the granting of the loan. The interest rate is 12% per annum compounded monthly. Calculate the monthly payments.

2. Mary takes out a ten-year loan of R150 000. She repays the loan by means of equal monthly payments starting seven months after the granting of the loan. The interest rate is 15% per annum compounded monthly. Calculate the monthly payments.

3. Thirty-two semi-annual payments of R6 000 are made in order to repay a loan. The payments start in two years’ time. Interest is 18.6% per annum compounded semi-annually. Find the size of the loan.

4. A loan of R120 454 is repaid by means of fourteen equal monthly payments, starting four years after the granting of the loan. Interest is 15% per annum compounded monthly. Find the value of the payments.

5. A loan is repaid, starting in five years’ time, by means of 12 quarterly payments of R7 000. What is the amount of the loan, if the interest is 24% per annum compounded quarterly?

6. Roxanne borrows R200 000 from a bank and repays the loan by means of monthly payments of R7 000, starting one month after the granting of the loan. Interest is fixed at 15% per annum compounded monthly.

(a) How many payments of R7 000 will be made and what will the final payment be?

(b) How long is the savings period?

7. How long will it take to repay a loan of R400 000 if the first quarterly payment of R17 000 is made three months after the granting of the loan and the interest rate is 16% per annum compounded quarterly?

8. A loan of R300 000 is to be repaid by means of monthly payments of R1 000, starting one month after the granting of the loan. Interest is fixed at 18% per annum compounded monthly. Determine whether or not the bank would grant this loan under the given conditions.

COMPARING INVESTMENTS AND LOANS

EXAMPLE 21

You decide to take out a bank loan of R800 000. Repayments are made on a monthly basis starting one month after the granting of the loan. You are offered three loan options:

Option A
Pay off the loan over twenty years at 18% per annum compounded monthly.

Option B
Pay off the loan over thirty years at 18% per annum compounded monthly.

Option C
Pay off the loan over thirty years at 15% per annum compounded monthly.

(a) Calculate the monthly amount payable for each option.

(b) Which option would be the best financially for you? Motivate your answer by using appropriate calculations.
Solutions

(a) Option A

Method 1
(Annuity formula)

\[ 800 000 = \frac{x \left[ 1 - (1,015)^{-240} \right]}{0,015} \]
\[ \therefore \frac{800 000 \times 0,015}{1 - (1,015)^{-240}} = x \]
\[ \therefore x = R12 346,49 \]

Method 2
(Geometric series)

\[ 800 000 = \frac{x(1,015)^{-1} \left[ (1,015)^{-240} - 1 \right]}{(0,015)^{-1} - 1} \]
\[ \therefore \frac{800 000 \times (0,015)^{-1} - 1}{(1,015)^{-1} \left[ (1,015)^{-240} - 1 \right]} = x \]
\[ \therefore x = R12 346,49 \]

Option B

Method 1
(Annuity formula)

\[ 800 000 = \frac{x \left[ 1 - (1,015)^{-360} \right]}{0,015} \]
\[ \therefore \frac{800 000 \times 0,015}{1 - (1,015)^{-360}} = x \]
\[ \therefore x = R12 056,68 \]

Method 2
(Geometric series)

\[ 800 000 = \frac{x(1,015)^{-1} \left[ (1,015)^{-360} - 1 \right]}{(0,015)^{-1} - 1} \]
\[ \therefore \frac{800 000 \times (0,015)^{-1} - 1}{(1,015)^{-1} \left[ (1,015)^{-360} - 1 \right]} = x \]
\[ \therefore x = R12 056,68 \]

Option C

Method 1
(Annuity formula)

\[ 800 000 = \frac{x \left[ 1 - (1,0125)^{-360} \right]}{0,0125} \]
\[ \therefore \frac{800 000 \times 0,0125}{1 - (1,0125)^{-360}} = x \]
\[ \therefore x = R10 115,55 \]

Method 2
(Geometric series)

\[ 800 000 = \frac{x(1,0125)^{-1} \left[ (1,0125)^{-360} - 1 \right]}{(0,0125)^{-1} - 1} \]
\[ \therefore \frac{800 000 \times (0,0125)^{-1} - 1}{(1,0125)^{-1} \left[ (1,0125)^{-360} - 1 \right]} = x \]
\[ \therefore x = R10 115,55 \]

b) The total amount paid over the twenty year period for Option A is:
R12 346,49 \times 240 = R2 963 157,60

The total amount paid over the thirty year period for Option B is:
R12 056,68 \times 360 = R4 340 404,80

The total amount paid over the thirty year period for Option C is:
R10 115,55 \times 360 = R3 641 598

Clearly, Option A is the better option because the total amount paid for the home will be the least of all the options.
EXERCISE 6

1. Determine which investment option is better. Payments start immediately.
   Option A  R2 000 invested on a monthly basis for 10 years at 18% per annum compounded monthly.
   Option B  R2 000 invested on a monthly basis for 15 years at 15% per annum compounded monthly.

2. Determine which loan option is better.
   Option A  A loan of R100 000 over 20 years at 15% per annum compounded monthly. Repayments are made on a monthly basis starting one month after the granting of the loan.
   Option B  A loan of R100 000 over 30 years at 12% per annum compounded monthly. Repayments are made on a monthly basis starting one month after the granting of the loan.

3. Chantelle wants to buy new furniture for her home costing R80 000. She has the following loan options available to her. Which option is better?
   Option A  A hire purchase loan over five years. The interest rate is 12% per annum simple interest.
   Option B  A reducing balance loan over five years. The interest rate is 14% per annum compounded monthly.

THE BALANCE OUTSTANDING ON A LOAN AT A GIVEN TIME

It is sometimes useful to calculate the balance still owed on a loan at a given time during the course of the loan.

EXAMPLE 22

James takes out a one year bank loan of R18 000 to pay for an expensive laptop. The interest rate is 18% per annum compounded monthly and monthly repayments of R1 650,24 are made starting one month after the granting of the loan.
(a) Calculate his balance outstanding after he has paid the sixth instalment.
(b) Calculate his balance outstanding after he has paid the ninth instalment.

Solutions

(a)

\[
\frac{0.18}{12} = 0.015
\]

At \( T_6 \), James has already paid six amounts of R1 650,24. He then still has six more to pay. Therefore the balance outstanding (B) after the sixth payment has been made can be seen as the present value of the payments yet to be made. The time line is now as follows:

\[
\begin{array}{cccccccccccccc}
T_0 & T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9 & T_{10} & T_{11} & T_{12} \\
\end{array}
\]
**Method 1** (Annuity formula)

\[
B = \frac{1650,24 \left[1 - (1,015)^{-6}\right]}{0,015}
\]

\[
\therefore B = \text{R}\, 9401,73
\]

If you refer to the table on page 82, you can verify this balance outstanding at the end of the sixth month.

**Method 2** (Geometric series)

The geometric series is:

\[
1650,24(1,015)^{-1} + 1650,24(1,015)^{-2} + \ldots + 1650,24(1,015)^{-6}
\]

\[
B = \frac{1650,24(1,015)^{-1}\left[(1,015)^{-6} - 1\right]}{(1,015)^{-1} - 1}
\]

\[
\therefore B = \text{R}\, 9401,73
\]

**Alternative method (optional)**

Grow the loan to \( T_6 \). Then determine the future value of the payments at \( T_6 \). Subtract to obtain the balance outstanding.

\[
B = 18\,000(1,015)^6 - \frac{1650,24\left[(1,015)^6 - 1\right]}{0,015} = 9401,72
\]

(The answer will be slightly different to the answer using the previous method).

(b) At \( T_9 \), James has already paid nine amounts of R1650,24. He then still has three more to pay. Therefore the balance outstanding (B), after the ninth payment has been made, can be seen as the present value of the payments yet to be made. The time line is now as follows:

\[\text{Method 1 (Annuity formula)}\]

\[
B = \frac{1650,24\left[1 - (1,015)^{-3}\right]}{0,015}
\]

\[
\therefore B = \text{R}\, 4805,83
\]

If you refer to the table on page 82, you can verify this balance outstanding at the end of the ninth month.

**Method 2** (Geometric series)

The geometric series is:

\[
1650,24(1,015)^{-1} + 1650,24(1,015)^{-2} + 1650,24(1,015)^{-3}
\]

\[
B = \frac{1650,24(1,015)^{-1}\left[(1,015)^{-3} - 1\right]}{(1,015)^{-1} - 1}
\]

\[
\therefore B = \text{R}\, 4805,83
\]
Alternative method (optional)
Grow the loan to \( T_9 \). Then determine the future value of the payments at \( T_9 \). Subtract to obtain the balance outstanding.

\[
B = 18\,000(1,015)^9 - \frac{1650,24(1,015)^9 - 1}{0,015} = 4805,83
\]

EXERCISE 7

1. Kevin takes out a bank loan to pay for his new car. He repays the loan by means of monthly payments of R3 000 for a period of six years starting one month after the granting of the loan. The interest rate is 18% per annum compounded monthly.
   (a) Calculate the purchase price of his new car.
   (b) Calculate the balance outstanding after the 20th payment.
   (c) Calculate the balance outstanding after the 60th payment.

2. Lerato takes out a bank loan and repays the loan by means of equal quarterly payments of R2 000, starting in three months time. The interest rate is 18% per annum compounded quarterly and the duration of the loan is ten years.
   (a) Calculate the amount borrowed.
   (b) Calculate the balance outstanding after the 30th payment has been made.

3. Sandy takes out a loan of R120 000 for home improvements. The loan is taken over four years at an interest rate of 12% per annum compounded monthly.
   (a) Calculate the monthly payments, if the first payment is made one month after the loan was granted.
   (b) Calculate the outstanding balance after the 20th payment has been made.

SINKING FUNDS

Many businesses will purchase equipment which will be used for a given period of time. After a number of years, this equipment is usually sold at scrap value and new upgraded equipment is bought. The business will often set up a savings plan at the time of purchasing the original equipment. This savings plan is a future value annuity which is called a sinking fund in the world of business.

EXAMPLE 23

A small business purchases a photocopying machine for R200 000. The photocopy machine depreciates in value at 20% per annum on a reducing balance. The small business wants to buy a new machine in five years time. A new machine will cost much more in the future and its cost will escalate at 16% per annum effective. The old machine will be sold at scrap value after five years. A sinking fund is set up immediately in order to save up for the new machine. The proceeds from the sale of the old machine will be used together with the sinking fund to buy the new machine. The small business will pay equal monthly amounts into the sinking fund and the interest earned is 18% per annum compounded monthly. The first payment
will be made immediately and the last payment will be made at the end of the five year period.

(a) Find the scrap value of the old machine.
(b) Find the cost of the new machine in five years time.
(c) Find the amount required in the sinking fund in five years time.
(d) Find the equal monthly payments made into the sinking fund.
(e) Suppose that the business decides to service the machine at the end of each year for the five year period. R3 000 will be withdrawn from the sinking fund at the end of each year starting one year after the original machine was bought.

(1) Calculate the reduced value of the sinking fund at the end of the five year period due to these withdrawals.

(2) Calculate the increased monthly payment into the sinking fund which will yield the original sinking fund amount as well as allow for withdrawals from the fund for the services.

Solutions

(a) \[ A = 200 \, 000(1 - 0,2)^5 = R65 \, 536 \]
(b) Cost of the new machine:
\[ C = 200 \, 000(1 + 0,16)^5 = R420 \, 068,33 \]
(c) Sinking fund + scrap value = cost of new machine
Sinking fund = cost of new machine – scrap value
Sinking fund = 420 068,33 – 65 536
Sinking fund = 354 532,33

(d) \[ x \times x \times x = 354 \, 532,33 \]
\[ 0,18 \over 12 = 0,015 \]

Method 1 (Annuity formula)
\[ x \left( \frac{(1,015)^61 - 1}{0,015} \right) = 354 \, 532,33 \]
\[ \therefore x = \frac{354 \, 532,33 \times 0,015}{(1,015)^61 - 1} \]
\[ \therefore x = R3593,55 \]

Method 2 (Geometric series)
The geometric series is: \[ x + x(1,015)^1 + x(1,015)^2 + \ldots + x(1,015)^{60} \]
\[ x \left( \frac{(1,015)^{61} - 1}{1,015 - 1} \right) = 354 \, 532,33 \]
\[ \therefore x = \frac{354 \, 532,33 \times 0,015}{(1,015)^{61} - 1} \]
\[ \therefore x = R3593,55 \]
(e)(1) The sinking fund will not only lose the five amounts of R3 000, it will also lose the interest earned on these amounts at the end of the five year period.

Future value of the withdrawals:

\[
3000 \left(1 + \frac{0.18}{12}\right)^{48} + 3000 \left(1 + \frac{0.18}{12}\right)^{36} + 3000 \left(1 + \frac{0.18}{12}\right)^{24} + 3000 \left(1 + \frac{0.18}{12}\right)^{12} + 3000
\]

= R22 133.22

The reduced value of the sinking fund will be:

R354 532.33 − R22 133.22 = R332 399.11

(e)(2) If we add R22 133.22 to the original sinking fund amount of R354 532.33, then it will be possible to not only receive the sinking fund amount of R354 532.33 at the end of the five year period, but also be able to make the service withdrawals at the end of each year for the five year period.

**Method 1**  (Annuity formula)

\[
354 532.33 + 22 133.22 = \frac{x \left[ (1.015)^{61} - 1 \right]}{0.015}
\]

\[
\therefore 376 665.55 = \frac{x \left[ (1.015)^{61} - 1 \right]}{0.015}
\]

\[
\therefore \underline{376 665.55 \times 0.015} = x
\]

\[
\therefore x = R3817,90
\]

**Method 2**  (Geometric series)

The geometric series is: \(x + x(1.015)^1 + x(1.015)^2 + \ldots + x(1.015)^{60}\)

\[
354 532.33 + 22 133.22 = \frac{x \left[ (1.015)^{61} - 1 \right]}{1.015 - 1}
\]

\[
\therefore 376 665.55 = \frac{x \left[ (1.015)^{61} - 1 \right]}{0.015}
\]

\[
\therefore \underline{376 665.55 \times 0.015} = x
\]

\[
\therefore x = R3817,90
\]
EXERCISE 8

1. A car wash business purchases large washing equipment for R140 000. The cost of the new is expected to rise by 18% per annum while the rate of depreciation is 20% per annum on the reducing-balance. The life span of the equipment is six years.
   (a) Find the scrap value of the original equipment.
   (b) Find the cost of new equipment in six years’ time.
   (c) Find the value of the sinking fund that will be required to purchase the new equipment in six years’ time, if the proceeds from the sale of the old equipment (at scrap value) will be used.
   (d) The business sets up a sinking fund to pay for the new equipment. Payments are to be made into an account paying 13.2% per annum compounded monthly. Find the monthly payments, if they are to commence one month after the purchase of the old equipment and cease at the end of the six year period.
   (e) Suppose that the business decides to service the equipment at the end of each year for the six year period. R4 000 will be withdrawn from the sinking fund at the end of each year starting one year after the original machine was bought.
      (1) Calculate the reduced value of the sinking fund at the end of the six year period due to these withdrawals.
      (2) Calculate the increased monthly payment into the sinking fund which will yield the original sinking fund amount as well as allow for withdrawals from the fund for the services.

2. Due to load shedding, a restaurant buys a large generator for R227 851. It depreciates at 23% per annum on a reducing-balance. A new generator is expected to appreciate in value at a rate of 17% per annum. A new generator will be purchased in five years’ time.
   (a) Find the scrap value of the old generator in five years’ time.
   (b) Find the cost of a new machine in five years time.
   (c) The restaurant will use the money received from the sale of the old machine (at scrap value) as part payment for the new one. The rest of the money will come from a sinking fund that was set up when the old generator was bought. Monthly payments, which started one month after the purchase of the old generator, have been paid into a sinking fund account paying 11.4% per annum compounded monthly. The payments will finish three months before the purchase of the new machine. Calculate the monthly payments into the sinking fund that will provide the required money for the purchasing of the new machine.
1. Tumelo buys a car for R100 000. He drives the car for four years and then decides to sell the car. Suppose that after four years of depreciation, the car is worth one quarter of its original value. The depreciation of his car is represented in the graph below.

(a) What is the value of the car at M on the graph?
(b) What is the value of the car at N on the graph?
(c) Based on the graph, what type of depreciation took place?
(d) Calculate the rate of depreciation as a percentage.

2. Edgar’s motor car costing R230 000 depreciated at a rate of 7% per annum on the reducing-balance method. Calculate how long it took for the car to depreciate to a value of R100 000 under these conditions.

3. R6 000 is invested at 9.6% per annum interest compounded quarterly. After how many years will the investment be worth R40 000?

4. Pat starts a five year savings plan. At the beginning of the month he deposits R1 400 into the account and makes a further deposit of R1400 at the end of that month. He then continues to make month-end payments of R1400 into the account for the five-year period (starting from his first deposit). The interest rate is 8% per annum compounded monthly.

(a) Calculate the future value of his investment at the end of the five-year period.
(b) If Pat leaves the accumulated amount in (a) in the account for a further six months, what will the value of his investment be at the end of the six-month period? Assume that the interest rate remains the same for the duration of the six-month period.

5. It is the 31st December 2012. Anna decides to start saving money and wants to save R700 000 in five years’ time by paying equal monthly amounts of Rx, starting in one months time (on 31st January 2013), into a savings account paying 9% per annum compounded monthly. The duration of the savings starts on the 31st December 2012, even though the first payment is not made on the 31st December 2012. Calculate the value of x.

6. Michael takes out a twenty-year retirement annuity. He makes monthly payments of R1 800 into the fund and the payments start immediately. The last payment of R1 800 is made two months before the annuity pays out. The interest rate for the annuity is 7,5% per annum compounded monthly. Calculate the future value of the annuity at the end of the twenty-year period.

7. R5 000 is invested each month, starting in one month’s time, into an account paying 6% per annum compounded monthly. The fund accumulates to R100 000. How many payments of R5 000 will be made?
8. How much can Belinda borrow from a bank if she repays the loan by means of equal monthly payments of R3 500, starting in one month’s time? The interest rate is 14% per annum compounded monthly and the duration of the loan is ten years.

9. Daphne wants to buy a house for R700 000. She puts down a deposit of R50 000 and takes out a loan for the balance at a rate of 18% per annum compounded monthly.
(a) How much money must Daphne borrow from the bank?
(b) Calculate the monthly payment if she wishes to settle the loan in fifteen years.
(c) Daphne won a lottery and wishes to settle the loan after the 50th payment. What is the outstanding balance?

10. Thembi takes out a loan of R150 000 for home improvements. The loan is taken over six years at an interest rate of 12% per annum compounded half-yearly. She repays the loan by means of annual payments of Rx.
(a) Convert the half-yearly rate to the annual effective rate.
(b) Calculate the annual payments, if the first payment is made one year after the granting of the loan.

11. Andrew wants to borrow money to buy a motorbike that costs R55 000 and plans to repay the full amount over a period of 4 years in monthly instalments. He is presented with TWO options:
**Option 1:**
The bank calculates what Andrew would owe if he borrows R55 000 for 4 years at a simple interest rate of 12.75% per annum and then pays that amount back in equal monthly instalments over 4 years.
**Option 2:**
He borrows R55 000 from the bank. He pays the bank back in equal instalments over 4 years, the first payment being made at the end of the first month. Compound interest at 20% per annum is charged on the reducing balance.
(a) If Andrew chooses Option 1, what will his monthly instalment be?
(b) Which option is the better option for Andrew? Justify your answer with appropriate calculations.
(c) What interest rate should replace 12.75% per annum in Option 1 so that there is no difference between the two options?

12. Ernest takes out a twenty year loan of R250 000. He repays the loan by means of equal monthly payments starting four months after the granting of the loan. The interest rate is 21% per annum compounded monthly.
(a) Calculate the amount owing three months after the loan was taken out by Ernest.
(b) Hence calculate the monthly repayments.

13. Lindiwe receives a bursary of R80 000 for her studies at university. She invests the money at a rate of 13.75% per annum compounded yearly. She decides to withdraw R25 000 at the end of each year for her studies, starting at the end of the first year. Determine for how many full years this investment will finance her studies.
14. A car that costs R130 000 is advertised in the following way: 'No deposit necessary and first payment due three months after date of purchase.' The interest rate quoted is 18% per annum compounded monthly.

(a) Calculate the amount owing two months after the purchase date, which is one month before the first monthly payment is due.

(b) Henry bought this car on 1 March 2009 and made his first payment on 1 June 2009. Thereafter he made another 53 equal payments on the first day of each month.

(1) Calculate his monthly repayments.

(2) Calculate the total of all Henry's repayments.

(c) Harry also bought a car for R130 000. He also took out a loan for R130 000, at an interest rate of 18% per annum compounded monthly. He also made 54 equal payments. However, he started payments one month after the purchase of the car. Calculate the total of all Harry's repayments.

(d) Calculate the difference between Henry’s and Harry’s total repayments.

SOME CHALLENGES

1. Jesse deposits R7 000 into an account paying 14% per annum compounded half-yearly. Six months later, she deposits R400 into the account. Six months after this, she deposits a further R400 into the account. She then continues to make half-yearly deposits of R400 into the account for a period of nine years from the deposit of R7 000. Calculate the value of her savings at the end of the savings period.

2. Josephine opened a savings account with a single deposit of R1 000 on the 1st April 2012. She then makes 18 monthly deposits of R700 at the end of every month. Her first payment is made on the 30th April 2012 and her last payment on 30th September 2013. The account earns interest at 15% per annum compounded monthly. Determine the amount that should be in her savings account immediately after her last deposit of R700 is made (i.e. on the 30th September 2013).

3. Sipho opens a savings account and immediately deposits R1 000 into the account. He continues to make monthly payments of R1 000 into the account for a period of three years. The interest rate for the first year is 18% per annum compounded monthly. Thereafter the interest rate changes to 19% per annum compounded monthly for the next two years. Calculate the value of his investment at the end of the savings period.

4. Michael opens a savings account and immediately deposits R1 000 into the account. He continues to make monthly payments of R1 000 into the account for a period of one year. The interest rate remains fixed at 18% per annum compounded monthly. Due to financial difficulty, he missed the 4th payment. Calculate the future value of his investment.

5. Kevin takes out a bank loan for R250 000. The interest rate charged by the bank is 18.5% per annum compounded monthly.

(a) What will his monthly repayment be if he pays the loan back over five years, starting FOUR months after the granting of the loan?

(b) Calculate the balance outstanding after the 25th repayment.
6. A father decided to buy a house for his family for R800 000. He agreed to pay monthly instalments of R10 000 on a loan which incurred interest at a rate of 14% per annum compounded monthly. The first payment was made at the end of the first month.
   (a) Show that the loan would be paid off in 234 months.
   (b) Suppose the father encountered unexpected expenses and was unable to pay any instalments at the end of the 120th, 121st, 122nd and 123rd months. At the end of the 124th month he increased his payment so as to still pay off the loan in 234 months by 111 equal monthly payments. Calculate the value of this new instalment. You may assume that the balance outstanding after the 119th payment has been made is R629 938.11.

7. Brenda takes out a twenty year loan of R400 000. She repays the loan by means of equal monthly payments starting one month after the granting of the loan. The interest rate is 18% per annum compounded monthly.
   (a) Calculate the monthly repayments.
   (b) Calculate the amount owed after the 3rd payment was made.
   (c) Due to financial difficulty, Brenda misses the 4th, 5th and 6th payments. Calculate her increased monthly payment which comes into effect from the 7th payment onwards.

8. Mr Brown has just finished paying off his twenty-year home loan which was R400 000. During the first five years the interest rate was 24% per annum compounded monthly. Thereafter, and for the rest of the term, the interest rate decreased to 18% per annum compounded monthly.
   (a) Calculate his initial monthly payment.
   (b) Calculate his balance outstanding at the end of December in the fifth year.
   (c) When the interest rate changed after five years, Mr Brown was able to pay a decreased monthly payment starting at the end of January in the sixth year. Calculate what this new repayment was.

9. Timothy buys furniture to the value of R10 000. He borrows the money on 1 February 2010 from a financial institution that charges interest at a rate of 9.5% per annum compounded monthly. Timothy agrees to pay monthly instalments of R450. The agreement of the loan allows Timothy to start paying these equal monthly instalments from 1 August 2010.
   (a) Calculate the total amount owing to the financial institution on 1 July 2010.
   (b) How many months will it take Timothy to pay back the loan?
   (c) What is the balance of the loan immediately after Timothy has made the 25th payment?
   (d) What is the final payment if it is made at the end of the month?
10. Vanessa borrows R1 250 000 from the bank in order to buy a new house. 
The interest rate is 14.4% per annum compounded monthly. The loan must 
be paid off in twenty years by means of monthly payments starting one 
month after the granting of the loan. 
(a) Find the monthly payments.
(b) If Vanessa doubles her monthly payment, calculate how long it will 
take her to repay the loan and what her final payment will be.

11. Jennifer bought a house for R620 000. She paid 35% cash and the balance 
was paid through a bank loan. The interest paid on the loan was 15% per 
annum compounded monthly.
(a) Calculate how much she would pay on a monthly basis if the loan is 
to be paid off over twenty years assuming that her first payment was 
made one month after the loan was granted.
(b) Calculate the balance outstanding at the end of the sixth year.
(c) At the end of the sixth year, the interest rate dropped to 12.25% per 
annum compounded monthly. Calculate her new monthly repayment 
one month after the interest rate dropped.

12. Jonathan opens a bank savings account on the last day of the year 2010 with 
a deposit of R2 500. He intends to make regular monthly payments of 
R1500 at the end of each month in 2011 and the first four months of 2012 in 
order to finance his overseas trip in May 2012. Interest for the entire period 
is calculated at 9% per annum compounded monthly. Assuming he achieves 
this aim, except for one missed payment at the end of February 2011, how 
much money will he have had in his account by the beginning of May 2012?

13. In eight years’ time a person wishes to pay cash for a car. He will require 
R350 000. He opens an investment account and earns 14% per annum 
compounded monthly. Payments start immediately and are made on a 
monthly basis.
(a) Find the monthly payments.
(b) At the end of the first year he pays an additional R15 000 into the 
account. One year later, he withdraws R20 000. Find the future 
value of the investment at the end of the eighth year. What is the 
shortfall amount?
PROJECT ON LEGAL AND ILLEGAL INVESTMENT SCHEMES

In this project, you will be required to investigate illegal investment scams which include Pyramid and Ponzi schemes. You will also focus on legal investment schemes which include network marketing, investments in insurance companies, banks and in the share market. The project aims to make you aware of the different investment opportunities available to you in South Africa and what schemes should be avoided at all costs.

In the Sunday Times (2012/07/14), the following article appeared:
A largely unknown Durban businesswoman has emerged as a central figure in one of South Africa's biggest pyramid schemes - thought to involve as much as R6-billion.
Nonhlanhla Doris Hadebe, 49, of Morningside, Durban, was found to have R77-million in her bank accounts when these were frozen by Reserve Bank investigators. Her R6.6-million fleet of luxury cars has also been seized as part of a local crackdown on Travel Ventures International (TVI) Express.
Papers filed in the High Court in Pietermaritzburg allege that Hadebe ran the South African operation of the international pyramid scheme.
TVI's operations have been described as the largest pyramid scheme operating in South Africa. The scheme, which also operates in Botswana, Swaziland and Namibia, has allegedly amassed R6-billion in the three years since it began operating from South Africa.
The Reserve Bank, through its supervision department, has probed 222 schemes in the past five years, with 28 cases opened last year.

[http://www.timeslive.co.za/sundaytimes/2012/07/14/ponzi-mama-s-r77m-bankbook]

**Task 1**

Discuss the difference between the following investment schemes.
(a) Pyramid schemes
(b) Ponzi scheme
(c) Multi-level marketing (network-marketing)
Explain why the first two are illegal and the third one is considered to be legal.

Useful internet links are provided below:
http://www.mathmotivation.com/money/pyramid-scheme.html
http://www.scamwatch.gov.au/content/index.phtml/tag/PyramidSchemes

**Task 2**

Explain how the following Pyramid schemes work and why they should be avoided:
(a) The chain letter scam
(b) The 8-ball model
Task 3  (Mathematical investigation)

Consider the following pyramid structure:

(a) You are at the top of the pyramid. How many people are below you in levels 1, 2, 3, 4 and 5?
(b) Write down the sequence formed if each term of the sequence represents the number of people in that level. What kind of sequence is formed?
(c) Using your knowledge of sequences and series, calculate the number of people in the following levels?
   (1) Level 8
   (2) Level 13
   (3) Level 20
   (4) Level 33

Explain why this pyramid is mathematically improbable and eventually breaks down.

(d) In pyramid below, the person recruiting does not get paid at all until he or she has recruited three levels worth of new members. You recruit two people at level 1. These two people recruit four at level 2, and these four recruit eight at level 3. When the eight are recruited, you receive the "participation fee" for all eight people of level 3. If the fee was R1000, then you would receive R8000. If sixteen people are then recruited at level 4, then the two people at level 1 would each receive R8000, as shown below.

(1) Write down the sequence formed starting below you and moving down towards the nth level.
(2) Hence write down the number of people in the last three levels in terms of n.
(3) Calculate the total number of people below you in the pyramid if there are n levels.
(e) Consider the following ratios: \( \frac{a}{x-1} \) and \( \frac{a}{x} \) where \( a > 0 \) and \( x > 1 \).

Investigate the relationship between these two ratios. Select different positive natural numbers for \( a \) and \( x \) and then compare the ratios and come up with a conclusion. This investigation will prove useful in (f).

(f) It has been shown that the people in the last three levels of the pyramid in lose all their money. Show by using your knowledge of geometric series and exponents that more than 88% of people in the pyramid will lose their money. Work with \( n \) and use the information obtained in (d) and (e).

Hint:
Work with the following ratio:
\[
\frac{\text{total number of people losing money}}{\text{total number of people in the pyramid including you}}
\]

**Task 4**
You have received a bonus of R10 000 from your employer in December and you wish to invest this money in a legal investment rather than a pyramid scheme. Investigate the best possible way to make your money grow over a period of 10 years by contacting different financial institutions. Ask them to provide you with investment information which informs you of the interest rate you will receive over the ten-year period.

Financial institutions that are legal but usually offer different interest rates include the following:
(1) Insurance companies
(2) Banks
(3) Share market

Determine the best option for you by contacting these institutions. Use your knowledge of the annuity formulae to verify the future value of your money at the interest rates given to you. Make an informed decision as to where you will invest your money. Consider all of the benefits available to you.

**MARKING RUBRIC FOR PROJECT**

**Task 1**  
(6 marks)

<table>
<thead>
<tr>
<th>Task</th>
<th>(a) Pyramid schemes</th>
<th>(b) Ponzi schemes</th>
<th>(c) Multi-level marketing</th>
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<tbody>
<tr>
<td>Score</td>
<td>Not answered or completely incorrect (0)</td>
<td>The scheme is explained but with a few minor inaccuracies (1)</td>
<td>The scheme is explained clearly and accurately (2)</td>
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### Task 2  (6 marks)

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<th>The scheme is explained clearly and accurately (3)</th>
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<tbody>
<tr>
<td>(a) Chain letter scam</td>
<td></td>
<td></td>
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<tr>
<td>(b) 8-ball model</td>
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**Total score for Task 2**

### Task 3  (20 marks)

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<tr>
<td>(a) Levels 1,2,3,4 and 5</td>
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<td>(b) Sequence formed</td>
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<tr>
<td>(c) Number per level</td>
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<tr>
<td>(d)(1) Sequence</td>
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</tr>
<tr>
<td>(d)(2) Last three levels</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>(d)(3) Total number in pyramid</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>(e) Ratios</td>
<td></td>
<td></td>
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<tr>
<td>(f) People losing money</td>
<td>No attempt is made or the question is completely incorrect with no indication of working with n or previous parts of the task. (0)</td>
<td>An attempt is made and there is some indication of working with n or previous parts of the task. The conclusion is not reached. (1-2)</td>
<td>An attempt is made and there is some indication of working with n or previous parts of the task. The conclusion is partially reached. (3-4)</td>
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**Total score for Task 3**
**Task 4**  (8 marks)

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<td>(a)</td>
<td>Insurance companies</td>
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<td>(b)</td>
<td>Banks</td>
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<td>(c)</td>
<td>Share market</td>
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<tr>
<td>(d)</td>
<td>Interest rates</td>
<td>Interest rates obtained (1)</td>
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<td></td>
<td>Interest rates not obtained (0)</td>
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<tr>
<td>(e)</td>
<td>Mathematical accuracy in determining the best investment option.</td>
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**Total score for Task 4**

**TOTAL MARKS: 40**
CHAPTER 4 – TRIGONOMETRY

Before studying Grade 12 Trigonometry, it is advisable to first do the following exercise which revises Grade 11 Trigonometry. Refer to your Grade 11 textbook and work through the concepts dealt with. Then attempt the following exercise.

REVISION EXERCISE  (REVISION OF GRADE 11 TRIGONOMETRY)

1. If \( \cos A = \frac{2\sqrt{6}}{5} \) and \( A \in [90^\circ; 360^\circ] \) calculate without the use of a calculator and with the aid of a diagram the value of \( 5\tan A \cdot \cos A \).

2. Point \( M(-5; -12) \) is a point on the Cartesian plane and reflex of \( \hat{XOM} = 0^\circ \).
   Calculate the following without the use of a calculator.
   (a) \( \cos \theta \)
   (b) \( 1 - \sin^2 \theta \)
   (c) \( \frac{\sin \theta}{\tan \theta} \)

3. Simplify the following:
   (a) \( \frac{\cos(180^\circ - \theta) \cdot \cos(90^\circ - \theta)}{\sin(90^\circ + \theta) \cdot \sin(\theta - 180^\circ)} \)
   (b) \( \frac{\cos(180^\circ + \theta) \cdot \tan(\theta)}{\sin(360^\circ - \theta) \cdot \tan(720^\circ + \theta)} \)
   (c) \( \frac{\cos(90^\circ - \theta)}{\sin(180^\circ - \theta)} - \sin^2(-\theta) \)
   (d) \( \frac{\cos^2(180 + x) - \tan 225^\circ}{\cos^2(90^\circ + x)} \)

4. If \( \tan 22^\circ = t \) write the following in terms of \( t \).
   (a) \( \tan 202^\circ \)
   (b) \( \tan 338^\circ \)
   (c) \( \tan(-22^\circ) \)
   (d) \( \tan 518^\circ \)
   (e) \( \cos 158^\circ \)
   (f) \( \sin 22^\circ \)

5. Simplify the following:
   (a) \( \sin 210^\circ - \tan 120^\circ \cdot \cos 330^\circ \)
   (b) \( \frac{\cos 410^\circ}{\sin 40^\circ} \)
   (c) \( \tan 240^\circ \cdot \cos(-210^\circ) - \sin^2 315^\circ \)

6. Prove that the following statements are identities:
   (a) \( (\sin \theta + \cos \theta)^2 = 1 + 2\sin \theta \cdot \cos \theta \)
   (b) \( \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta} \)
   (c) \( \frac{1}{1 - \cos \theta} + \frac{1}{1 + \cos \theta} = \frac{2}{\sin^2 \theta} \)
   (d) \( \tan(90^\circ - \theta) = \frac{1}{\tan \theta} \)

7. Find the general solution of each of the following equations:
   (a) \( \cos 2\theta = 0,4 \)
   (b) \( 5\sin \theta + 14 \cos \theta = 0 \)

8. Solve for \( x \) if \( \sin(x + 30^\circ) = -0,2 \) where \( x \in [-180^\circ; 0^\circ] \)
COMPOUND ANGLE FORMULAE

Compound angles involve the trigonometric ratio of the sum of two angles or the difference between two angles.

DISCOVERY EXERCISE

1. Given:
   (i) \( A = 60^\circ \) and \( B = 30^\circ \)
   (ii) \( A = 110^\circ \) and \( B = 50^\circ \)
   (iii) \( A = 225^\circ \) and \( B = 135^\circ \)

   Use a calculator to evaluate each of the following:
   (a) \( \cos(A - B) \)
   (b) \( \cos A \cos B \)
   (c) \( \cos A \cos B + \sin A \sin B \)

2. (a) What do you notice about the values of \( \cos(A - B) \) and \( \cos A \cos B \)?
   (b) What do you notice about the values of \( \cos(A - B) \) and \( \cos A \cos B + \sin A \sin B \)?

From the above discovery exercise it should be clear that \( \cos(A - B) \neq \cos A \cos B \)
and that \( \cos(A - B) = \cos A \cos B + \sin A \sin B \) for the values given.

Now we will prove that the identity \( \cos(A - B) = \cos A \cos B + \sin A \sin B \) is true
for all values of \( A \) and \( B \). The proof is not for examination purposes.

Proof

Let \( P(\cos \alpha ; \sin \alpha) \) and \( Q(\cos \beta ; \sin \beta) \)
be any two points on the unit circle \( O \) with radius 1.

If \( \angle P\hat{O}X = \alpha \) and \( \angle Q\hat{O}X = \beta \) then
\( \angle P\hat{O}Q = \alpha - \beta \)

From the cosine rule:
\[ PQ^2 = l^2 + l^2 - 2(l)(l)\cos(\alpha - \beta) \]
\[ \therefore PQ^2 = 2 - 2\cos(\alpha - \beta) \] \( \ldots A \)

From the distance formula:
\[ PQ^2 = (x_P - x_Q)^2 + (y_P - y_Q)^2 \]
\[ \therefore PQ^2 = (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 \]
\[ \therefore PQ^2 = \cos^2 \alpha - 2\cos \alpha \cos \beta + \cos^2 \beta + \sin^2 \alpha - 2\sin \alpha \sin \beta + \sin^2 \beta \]
\[ \therefore PQ^2 = \cos^2 \alpha + \sin^2 \alpha + \cos^2 \beta + \sin^2 \beta - 2\cos \alpha \cos \beta - 2\sin \alpha \sin \beta \]
\[ \therefore PQ^2 = 1 + 1 - 2\cos \alpha \cos \beta - 2\sin \alpha \sin \beta \]
\[ \therefore PQ^2 = 2 - 2\cos \alpha \cos \beta - 2\sin \alpha \sin \beta \] \( \ldots B \)

Now we can equate the two equations \( (A \) and \( B) \)
\[ 2 - 2\cos(\alpha - \beta) = 2 - 2\cos \alpha \cos \beta - 2\sin \alpha \sin \beta \] \( (A = B) \)
\[ \therefore -2\cos(\alpha - \beta) = -2\cos \alpha \cos \beta - 2\sin \alpha \sin \beta \]
\[ \therefore \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \] \( +(-2) \)
Using the compound angle formula \( \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \) and suitable reduction formulae we can show that:

\[
\cos(A + B) = \cos A \cos B - \sin A \sin B
\]

\[
\sin(A - B) = \sin A \cos B - \cos A \sin B
\]

\[
\sin(A + B) = \sin A \cos B + \cos A \sin B
\]

The derivation of these identities is left as an exercise for you to do.

**EXAMPLE 1**

(a) Expand each of the following:

1. \( \cos(X - Y) \)

2. \( \sin(A - 20^\circ) \)

3. \( \sin(2\alpha + 45^\circ) \)

(b) Show that \( \cos(90^\circ + \theta) = -\sin \theta \) by using the appropriate compound angle formula.

**Solutions**

(a)  
1. \( \cos(X - Y) = \cos X \cos Y + \sin X \sin Y \)

2. \( \sin(A - 20^\circ) = \sin A \cos 20^\circ - \cos A \sin 20^\circ \)

3. \( \sin(2\alpha + 45^\circ) = \sin 2\alpha \cos 45^\circ + \cos 2\alpha \sin 45^\circ \)

\[
= \sin 2\alpha \left( \frac{\sqrt{2}}{2} \right) + \cos 2\alpha \left( \frac{\sqrt{2}}{2} \right)
\]

\[
= \left( \frac{\sqrt{2}}{2} \right)(\sin 2\alpha + \cos 2\alpha) = \frac{\sqrt{2}(\sin 2\alpha + \cos 2\alpha)}{2}
\]

(b) \( \cos(90^\circ + \theta) = \cos 90^\circ \cos \theta - \sin 90^\circ \sin \theta \)

\[= (0) \cos \theta - (1) \sin \theta\]

\[= 0 - \sin \theta\]

\[= -\sin \theta\]

**EXAMPLE 2**

(a) Express the following as a single trigonometric ratio:

1. \( \sin 2\theta \cos \theta + \cos 2\theta \sin \theta \)

2. \( \cos 70^\circ \cos x + \sin 70^\circ \sin x \)

3. \( \cos x \sin 3x - \cos 3x \sin x \)

4. \( \sin 30^\circ \sin 20^\circ - \cos 30^\circ \cos 20^\circ \)

(b) Without using a calculator, calculate the value of each of the following:

1. \( \cos 320^\circ \cos 20^\circ - \sin 140^\circ \sin 200^\circ \)

2. \( \cos 10^\circ \sin 160^\circ - \sin 10^\circ \sin 110^\circ \)

**Solutions**

(a)  
1. \( \sin 2\theta \cos \theta + \cos 2\theta \sin \theta = \sin(2\theta + \theta) = \sin 3\theta \)

2. \( \cos 70^\circ \cos x + \sin 70^\circ \sin x = \cos(70^\circ - x) \)

3. \( \cos x \sin 3x - \cos 3x \sin x \)

\[= \sin 3x \cos x - \cos 3x \sin x \quad (\text{rewrite } \cos x \sin 3x \text{ as } \sin 3x \cos x) \]

\[= \sin (3x - x) \quad (\text{sin } A \cos B - \cos A \sin B = \sin (A - B)) \]

\[= \sin 2x \]
(4) \[ \sin 30 \cdot \sin 20 - \cos 30 \cdot \cos 20 \]
\[ = - (\cos 30 \cdot \cos 20 - \sin 30 \cdot \sin 20) \]
\[ = - \cos(30 + 20) = - \cos 50 \]
\[ (\cos A \cdot \cos B - \sin A \cdot \sin B = \cos(A + B)) \]

(b) (1) \[ \cos 320^\circ \cdot \cos 20^\circ + \sin 140^\circ \cdot \sin 200^\circ \]
\[ = \cos(360^\circ - 40^\circ) \cdot \cos 20^\circ + \sin(180^\circ - 40^\circ) \cdot \sin(180^\circ + 20^\circ) \]
\[ = \cos 40^\circ \cdot \cos 20^\circ + \sin 40^\circ \cdot (- \sin 20^\circ) \]
\[ = \cos 40^\circ \cdot \cos 20^\circ - \sin 40^\circ \cdot \sin 20^\circ \]
\[ = \cos(40^\circ + 20^\circ) \]
\[ = \cos 60^\circ = \frac{1}{2} \]

(2) \[ \cos 10^\circ \cdot \sin 160^\circ - \sin 10^\circ \cdot \sin 110^\circ \]
\[ = \cos 10^\circ \cdot \sin(180^\circ - 20^\circ) - \sin 10^\circ \cdot \sin(180^\circ - 70^\circ) \]
\[ = \cos 10^\circ \cdot \sin 20^\circ - \sin 10^\circ \cdot \sin 70^\circ \]
\[ = \cos 10^\circ \cdot \sin 20^\circ - \sin 10^\circ \cdot \sin(90^\circ - 20^\circ) \]
\[ = \cos 10^\circ \cdot \sin 20^\circ - \sin 10^\circ \cdot \cos 20^\circ \]
\[ = \sin(20^\circ - 10^\circ) \]
\[ = \sin 10^\circ \]

Summary of formulae:

<table>
<thead>
<tr>
<th>Compound angle formulae</th>
<th>cos(A - B) = cos A \cdot cos B + sin A \cdot sin B</th>
<th>cos(A + B) = cos A \cdot cos B - sin A \cdot sin B</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin(A - B) = sin A \cdot cos B - cos A \cdot sin B</td>
<td>sin(A + B) = sin A \cdot cos B + cos A \cdot sin B</td>
<td></td>
</tr>
</tbody>
</table>

EXERCISE 1

1. Expand each of the following using the compound angle formulae:
   (a) \[ \cos(x - 40^\circ) \]
   (b) \[ \sin(x - 50^\circ) \]
   (c) \[ \cos(2x + y) \]
   (d) \[ \sin(10^\circ + B) \]
   (e) \[ \cos(60^\circ - A) \]
   (f) \[ \sin(45^\circ - x) \]
   (g) \[ \sin(A + A) \]
   (h) \[ \cos(A + A) \]

2. Use the compound angle formulae to simplify each expression to one term only:
   (a) \[ \cos 30^\circ \cdot \cos \theta + \sin 30^\circ \cdot \sin \theta \]
   (b) \[ \cos 2x \cdot \cos 3x - \sin 2x \cdot \sin 3x \]
   (c) \[ \sin 4P \cdot \cos 2P - \cos 4P \cdot \sin 2P \]
   (d) \[ \sin 10^\circ \cdot \cos 40^\circ + \cos 10^\circ \cdot \sin 40^\circ \]
   (e) \[ \sin 3x \cdot \cos x - \cos 3x \cdot \sin x \]
   (f) \[ \cos 2x \cdot \sin 3x - \cos 3x \cdot \sin 2x \]
   (g) \[ \sin 4\theta \cdot \sin 50^\circ - \cos 50^\circ \cdot \cos 4\theta \]

3. Calculate each of the following without the use of a calculator:
   (a) \[ \cos 80^\circ \cdot \cos 10^\circ - \sin 80^\circ \cdot \sin 10^\circ \]
   (b) \[ \cos 70^\circ \cdot \cos 40^\circ + \sin 70^\circ \cdot \sin 140^\circ \]
   (c) \[ \sin 280^\circ \cdot \cos 160^\circ - \cos 100^\circ \cdot \sin 200^\circ \]
   (d) \[ \cos 265^\circ \cdot \sin 355^\circ - \sin 85^\circ \cdot \cos 175^\circ \]
   (e) \[ \cos 65^\circ \cdot \cos 295^\circ - \sin 115^\circ \cdot \cos 205^\circ \]
   (f) \[ \cos 50^\circ \cdot \sin 260^\circ + \cos 10^\circ \cdot \sin 140^\circ \]
4. Show that the following statements are identities using your compound-angle formulae:
   (a) \( \sin(180° + \theta) = -\sin \theta \)  
   (b) \( \cos(A - B) - \cos(A + B) = 2\sin A \cdot \sin B \)

It is important to note that you don’t have to use compound angle formulae to simplify function values of angles \( 180° \pm \theta, 360° \pm \theta \) or \( 90° \pm \theta \).

**EXAMPLE 3**

(a) Show that \( \cos(60° + \theta) - \cos(60° - \theta) = -\sqrt{3} \sin \theta \)
(b) Hence evaluate \( \cos105° - \cos15° \) without using a calculator.

**Solutions**
(a) \[
\begin{align*}
\cos 60° \cdot \cos \theta - \sin 60° \cdot \sin \theta &= \left(\frac{1}{2}\right) \cdot \cos \theta - \left(\frac{\sqrt{3}}{2}\right) \cdot \sin \theta \\
&= \frac{\cos \theta}{2} - \frac{\sqrt{3} \sin \theta}{2} - \frac{\cos \theta - \sqrt{3} \sin \theta}{2} \\
&= \frac{-2\sqrt{3} \sin \theta}{2} = -\sqrt{3} \sin \theta
\end{align*}
\]

(b) The word hence implies that you must use what you have previously shown or proven. In this example, rewrite the angles \( 105° \) and \( 15° \) as \( 60° + ... \) and \( 60° - ... \) respectively.
\[
\therefore \cos105° - \cos15° = \cos(60° + 45°) - \cos(60° - 45°)
\]
It has been shown that \( \cos(60° + \theta) - \cos(60° - \theta) = -\sqrt{3} \sin \theta \)
\[
\therefore \theta = 45°
\]
\[
\therefore \cos(60° + 45°) - \cos(60° - 45°) = -\sqrt{3} \sin 45° = -\sqrt{3} \left(\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{6}}{2}
\]

**EXAMPLE 4**
Calculate \( \sin75° \) without the use of a calculator.

**Solution**
\[
\sin75° = \sin(45° + 30°)
\]
\[
\therefore \sin75° = \sin45° \cdot \cos30° + \cos45° \cdot \sin30°
\]
\[
\therefore \sin75° = \left(\frac{\sqrt{2}}{2}\right) \cdot \left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right) \cdot \left(\frac{1}{2}\right) = \frac{\sqrt{6} + \sqrt{2}}{4} = \frac{\sqrt{6} + \sqrt{2}}{4}
\]
**EXERCISE 2**

1. Prove the following:
   (a) \( \cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta \)
   (b) \( \sin(\theta + 30^\circ) - \sin(\theta - 30^\circ) = \cos\theta \)
   (c) \( \cos(45^\circ - \theta) - \sin(45^\circ - \theta) = \sqrt{2}\sin\theta \)
   (d) \( \sqrt{3}\sin(\theta + 60^\circ) - \sin(\theta + 30^\circ) = \cos\theta \)
   (e) \( \sin(\alpha - 60^\circ) + \cos(\alpha - 30^\circ) = \sin\alpha \)

2. Calculate the following without the use of a calculator:
   (a) \( \cos 75^\circ \) (b) \( \sin 15^\circ \) (c) \( \cos 165^\circ \)

3. Triangle ABC is given.
   (a) Determine angle A in terms of B and C.
   (b) Hence, show that:
       \( \sin A\sin B\cos C\cos B - \sin C = \theta \)

4. Show that if \( \sin(\theta + 60^\circ) = 2\sin\theta \) then \( \tan\theta = \frac{\sqrt{3}}{3} \)

**DOUBLE ANGLE FORMULAE**

If the expression \( \sin 2A \) is expanded using a compound angle identity, then a new identity is formed:
\[
\sin 2A = \sin(A + A)
\]
\[
= \sin A\cos A + \cos A\sin A
\]
\[
= 2\sin A\cos A
\]
\[
\therefore \sin 2A = 2\sin A\cos A
\]

If the expression \( \cos 2A \) is expanded using a compound angle identity, then new identities are formed:
\[
\cos 2A = \cos(A + A)
\]
\[
= \cos A\cos A - \sin A\sin A
\]
\[
\therefore \cos 2A = \cos^2 A - \sin^2 A
\]

From the identity \( \cos^2 A + \sin^2 A = 1 \) we can deduce that:
\[
\cos 2A = 1 - \sin^2 A \text{ and } \sin^2 A = 1 - \cos^2 A .
\]
\[
\therefore \cos 2A = \cos^2 A - \sin^2 A \text{ or } \cos 2A = \cos^2 A - \sin^2 A
\]
\[
\therefore \cos 2A = (1 - \sin^2 A) - \sin^2 A \text{ or } \cos 2A = \cos^2 A - (1 - \cos^2 A)
\]
\[
\therefore \cos 2A = 1 - 2\sin^2 A \text{ or } \cos 2A = \cos^2 A - 1 + \cos^2 A
\]
\[
\therefore \cos 2A = 2\cos^2 A - 1
Summary of double angle formulae

\[
\begin{align*}
\sin 2A &= 2\sin A \cos A \\
\cos 2A &= \cos^2 A - \sin^2 A \\
\end{align*}
\]

EXAMPLE 5

(a) Use the double angle formulae of sine to expand the following:
   (1) \(\sin 4A\)  
   (2) \(\sin 50^\circ\)  
   (3) \(\sin A\)

(b) Use the double angle formulae of cosine to expand the following:
   (1) \(\cos 6A\)  
   (2) \(\cos 80^\circ\)  
   (3) \(\cos A\)

Solutions

(a) (1) \(\sin 4A = \sin[2(2A)] = 2\sin 2A \cos 2A\)  
   (2) \(\sin 50^\circ = \sin[2(25^\circ)]\)  
   (3) \(\sin A = \sin \left[2\left(\frac{A}{2}\right)\right] = 2\sin \frac{A}{2} \cos \frac{A}{2}\)

(b) (1) \(\cos 6A = \cos[2(3A)] = \cos^2 3A - \sin^2 3A\)  
   (2) \(\cos 80^\circ = \cos[2(40^\circ)]\)  
   (3) \(\cos A = \cos \left[2\left(\frac{A}{2}\right)\right] = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}\)

EXAMPLE 6

(a) Write as the sine of a single angle:
   (1) \(2\sin 30^\circ \cos 30^\circ\)  
   (2) \(2\sin \frac{A}{2} \cos \frac{A}{2}\)  
   (3) \(4\sin x \cos x\)

(b) Write as the cosine of a single angle:
   (1) \(\cos^2 20^\circ - \sin^2 20^\circ\)  
   (2) \(1 - 2\sin^2 20^\circ\)  
   (3) \(\sin^2 \theta - \cos^2 \theta\)

Solutions

(a) (1) \(2\sin 30^\circ \cos 30^\circ = \sin[2(30^\circ)] = \sin 60^\circ\)  
   (2) \(2\sin \frac{A}{2} \cos \frac{A}{2} = \sin \left[2\left(\frac{A}{2}\right)\right] = \sin A\)

(b) (1) \(4\sin x \cos x = 2 \times 2\sin x \cos x = 2\sin 2x\)  
   (2) \(4\sin x \cos x \neq \sin 4x\)
(b) (1) \( \cos^2 \theta - \sin^2 \theta = \cos \left[ 2 \left( \frac{\theta}{2} \right) \right] = \cos 40 \)
(2) \( 1 - 2 \sin^2 2\theta = \cos \left[ 2 \left( 2 \theta \right) \right] = \cos 40 \)
(3) \( \sin^2 \theta - \cos^2 \theta \)
\[ = -(\cos^2 \theta - \sin^2 \theta) \]
\[ = -\cos 20 \]

**EXAMPLE 7**

Calculate the following without using a calculator:

(a) \( 2 \sin 15^\circ \cdot \cos 15^\circ \)  
(b) \( 1 - 2 \cos^2 22.5^\circ \)  
(c) \( (\cos 15^\circ + \sin 15^\circ)^2 \)

**Solutions**

(a) \( 2 \sin 15^\circ \cdot \cos 15^\circ \)
\[ = \sin \left[ 2 \left( 15^\circ \right) \right] \]
\[ = \sin 30^\circ \]
\[ = \frac{1}{2} \]

(b) \( 1 - 2 \cos^2 22.5^\circ \)
\[ = -(2 \cos^2 22.5^\circ - 1) \]
\[ = -\cos(2(22.5^\circ)) \]
\[ = -\cos 45^\circ \]
\[ = -\frac{\sqrt{2}}{2} \]

(c) \( (\cos 15^\circ + \sin 15^\circ)^2 \)
\[ = \cos^2 15^\circ + 2 \cos 15^\circ \cdot \sin 15^\circ + \sin^2 15^\circ \]
\[ = \cos^2 15^\circ + 2 \cos 15^\circ \cdot \sin 15^\circ + 2 \sin 15^\circ \cdot \cos 15^\circ \]
\[ = 1 + \sin \left[ 2 \left( 15^\circ \right) \right] \]
\[ = 1 + \sin 30^\circ \]
\[ = 1 + \frac{1}{2} = \frac{3}{2} \]

**EXERCISE 3**

1. Expand the following using double angle formulae:
   (a) \( \cos 4x \)  
   (b) \( \sin 6\theta \)  
   (c) \( \cos 10\theta \)  
   (d) \( \sin 50^\circ \)  
   (e) \( \cos 20^\circ \)  
   (f) \( \sin 70^\circ \)  
   (g) \( 4 \sin 44^\circ \)  
   (h) \( 2 \cos 22^\circ \)  
   (i) \( -\cos 86^\circ \)
2. Write as a single trigonometric function:
   (a) \( 2 \sin x \cdot \cos x \)  
   (b) \( 2 \sin 3A \cdot \cos 3A \)  
   (c) \( 2 \sin 20^\circ \cdot \cos 20^\circ \)  
   (d) \( 2 \sin 35^\circ \cdot \cos 35^\circ \)  
   (e) \( 4 \sin \theta \cdot \cos \theta \)  
   (f) \( \cos^2 30^\circ - \sin^2 30^\circ \)  
   (g) \( 2 \cos^2 \theta - 1 \)  
   (h) \( 1 - 2 \sin^2 \frac{\alpha}{2} \)  
   (i) \( \sin^2 42^\circ - \cos^2 42^\circ \)  
   (j) \( 1 - 2 \cos^2 22^\circ \)  
   (k) \( (\cos 2\theta + \sin 2\theta)(\cos 2\theta - \sin 2\theta) \)  
   (l) \( \sin \theta \cos \theta \)
3. Simplify the following without using a calculator:
   (a) \(2 \sin 22.5\degree \cos 22.5\degree\)  
   (b) \(2 \sin 75\degree \cos 75\degree\)  
   (c) \(2 \sin(-15\degree) \cos(-15\degree)\)  
   (d) \(4 \sin 105\degree \cos 105\degree\)  
   (e) \(\sin 165\degree \cos 165\degree\)  
   (f) \(\cos^2 15\degree - \sin^2 15\degree\)  
   (g) \(\cos^2 15\degree + \sin^2 15\degree\)  
   (h) \((\cos 15\degree - \sin 15\degree)^2\)  
   (i) \(1 - 2 \sin^2 15\degree\)  
   (j) \(2 \cos^2 75\degree - 1\)  
   (k) \(2 \sin^2 22.5\degree - 1\)  
   (l) \(1 - 2 \cos^2 105\degree\)

4. Show that \(\sin 80\degree = 8 \sin \theta \cos \theta \cos 20\degree \cos 40\degree\)

**EXAMPLE 8**

(a) Evaluate:  
   (1) \(\frac{\sin 35\degree \cos 35\degree}{\tan 225\degree \cos 200\degree}\)  
   (2) \(\frac{\sin 36\degree \cos 36\degree}{\sin 12\degree - \cos 12\degree}\)

(b) Rewrite \(\sin 3\alpha\) in terms of \(\sin \alpha\):

**Solutions**

(1) \(\frac{\sin 35\degree \cos 35\degree}{\tan(180\degree + 45\degree) \cos(180\degree + 20\degree)}\)  
   \(= \frac{\sin 35\degree \cos 35\degree}{\tan 45\degree (-\cos 20\degree)}\)  
   \(= \frac{\sin 35\degree \cos 35\degree}{(1)(-\sin 70\degree)}\)  
   \(= \frac{\sin 35\degree \cos 35\degree}{(-\sin 2(35\degree))}\)  
   \(= \frac{\sin 35\degree \cos 35\degree}{-2 \sin 35\degree \cos 35\degree}\)  
   \(= -\frac{1}{2}\)  

(reduce angles to acute angles)

(2) \(\frac{\sin 36\degree \cos 12\degree - \cos 36\degree \sin 12\degree}{\sin 12\degree \cos 12\degree - \sin 12\degree \cos 12\degree}\)  
   \(= \frac{\sin 36\degree \cos 12\degree - \cos 36\degree \sin 12\degree}{\sin 12\degree \cos 12\degree}\)  
   \(= \frac{\sin(36\degree - 12\degree)}{\sin 12\degree \cos 12\degree}\)  
   \(= \frac{\sin 24\degree}{\sin 12\degree \cos 12\degree}\)  
   \(= \frac{\sin 2(12\degree)}{\sin 12\degree \cos 12\degree}\)  
   \(= \frac{2 \sin 12\degree \cos 12\degree}{\sin 12\degree \cos 12\degree} = 2\)  

(LCD: \(\sin 12\degree \cos 12\degree\))

(\(\sin A \cos B - \cos A \sin B = \sin (A - B)\))

(24\degree is double 12\degree)
(b) \( \sin 3\alpha = \sin (2\alpha + \alpha) \)
\[ = \sin 2\alpha \cdot \cos \alpha + \cos 2\alpha \cdot \sin \alpha \]
\[ = (2 \sin \alpha \cdot \cos \alpha) \cdot \cos \alpha + (1 - 2 \sin^2 \alpha) \cdot \sin \alpha \]
\[ = 2 \sin \alpha \cdot \cos^2 \alpha + \sin \alpha - 2 \sin^3 \alpha \]
\[ = 2 \sin \alpha \cdot (1 - \sin^2 \alpha) + \sin \alpha - 2 \sin^3 \alpha \]
\[ = 2 \sin \alpha - 2 \sin^3 \alpha + \sin \alpha - 2 \sin^3 \alpha \]
\[ = 3 \sin \alpha - 4 \sin^3 \alpha \]

The reason for choosing the identity \( \cos 2\alpha = 1 - 2 \sin^2 \alpha \) is because it is required to write the expression in terms of \( \sin \alpha \).

**Summary**

**Compound angle formulae**

\[
\begin{align*}
\cos (A - B) &= \cos A \cdot \cos B + \sin A \cdot \sin B \\
\sin (A - B) &= \sin A \cdot \cos B - \cos A \cdot \sin B
\end{align*}
\]

**Double angle formulae**

\[
\begin{align*}
\sin 2A &= 2 \sin A \cdot \cos A \\
\cos 2A &= \frac{\cos^2 A - \sin^2 A}{2} \\
&= \frac{1 - 2 \sin^2 A}{2}
\end{align*}
\]

**EXERCISE 4**

1. Evaluate without using a calculator:
   (a) \( \frac{\sin 40^\circ \cdot \cos 40^\circ}{\sin 100^\circ} \)
   (b) \( \frac{\cos^2 22.5^\circ + \sin^2 22.5^\circ}{\cos^2 22.5^\circ - \sin^2 22.5^\circ} \)
   (c) \( 2 \sin 105^\circ \cdot \cos 255^\circ \)
   (d) \( 2 \cos 15^\circ \cdot \cos 75^\circ \)
   (e) \( 1 - 4 \sin^2 30^\circ \)
   (f) \( \cos 40^\circ + 2 \sin^2 200^\circ \)
   (g) \( \tan 135^\circ + 2 \cos^2 15^\circ \)
   (h) \( \tan 15^\circ + \frac{\cos 15^\circ}{\sin 15^\circ} \)
   (i) \( \cos^2 15^\circ + \sin 22.5^\circ \cdot \cos 22.5^\circ - \sin^2 15^\circ \)

2. Evaluate without using a calculator:
   \( \frac{\sin 72^\circ}{\sin 24^\circ} - \frac{\cos 72^\circ}{\cos 24^\circ} \)

3. Calculate: \( \sin 110^\circ + \cos 70^\circ \cdot \tan 190^\circ \)

4. Rewrite \( \cos 3A \) in terms of \( \cos A \)

5. (a) Show that \( \sin(45^\circ - \alpha) = \frac{\sqrt{2} (\cos \alpha - \sin \alpha)}{2} \)
(b) Hence prove that \( \sin 2\alpha + 2 \sin^2 (45^\circ - \alpha) = 1 \)
EXAMPLES INVOLVING DIAGRAMS AND PYTHAGORAS

EXAMPLE 9

If \( \tan \theta = \frac{3}{4} \) and \( \theta \in (180^\circ ; 360^\circ) \) calculate without the use of a calculator and with the aid of a diagram the value of: (a) \( \sin 2\theta \) (b) \( \cos 2\theta \) (c) \( \tan 2\theta \)

Solutions

Given: \( \tan \theta = \frac{3}{4} \)  
\( \tan \theta \) is positive and therefore the terminal arm will lie in the **first or third** quadrant.

But with \( \theta \in (180^\circ ; 360^\circ) \), the terminal arm will lie in the **third** quadrant.

\[ x^2 + y^2 = r^2 \ldots \text{Pythagoras.} \]
\[ x = -4 \] and \( y = -3 \) (because of the quadrant)
\[ \therefore (-4)^2 + (-3)^2 = r^2 \]
\[ 16 + 9 = r^2 \]
\[ \therefore 25 = r^2 \]
\[ \therefore r = \pm 5 \]

But \( r \) is always positive
\[ \therefore r = 5 \]

(a) \( \sin 2\theta = 2\sin \theta \cos \theta \)  
\[ \therefore \sin 2\theta = 2 \left( \frac{-3}{5} \right) \left( \frac{-4}{5} \right) \]
\[ \therefore \sin 2\theta = \frac{-24}{25} \]

(b) \( \cos 2\theta = 2\cos^2 \theta - 1 \)  
\[ \therefore \cos 2\theta = 2 \left( \frac{-4}{5} \right)^2 - 1 \]
\[ \therefore \cos 2\theta = \frac{24}{25} - 1 = \frac{7}{25} \]

(c) \( \tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} \)  
\[ \therefore \tan 2\theta = \frac{\frac{-24}{25}}{\frac{7}{25}} = \frac{24}{7} \]

EXAMPLE 10

If \( \sin \alpha = \frac{3}{5} \) with \( \alpha \in (90^\circ ; 270^\circ) \) and \( \cos \beta = \frac{12}{13} \) with \( \beta \in (180^\circ ; 360^\circ) \) calculate without the use of a calculator and with the aid of a diagram the value of

1. \( \sin (\alpha - \beta) \)  
2. \( \cos (\alpha + \beta) \)  
3. \( \cos (90^\circ - \alpha + \beta) \)

Solution

We will consider each trigonometric equation separately.

Given: \( \sin \alpha = \frac{3}{5} \)  
\( \sin \alpha \) is positive in the 1st and 2nd quadrant and \( \alpha \in (90^\circ ; 270^\circ) \).

The terminal arm lies in the 2nd quadrant.

Given: \( \cos \beta = \frac{12}{13} \)  
\( \cos \beta \) is positive in the 1st and 4th quadrant and \( \beta \in (180^\circ ; 360^\circ) \).

The terminal arm lies in the 4th quadrant.
\[(x; 3)\]
\[
\begin{align*}
x^2 + y^2 &= r^2 \quad \text{Pythagoras.} \\
y &= 3 \quad \text{and} \quad r = 5 \\
x^2 &= r^2 - y^2 \\
\therefore x^2 &= (5)^2 - (3)^2 \\
\therefore x^2 &= 16 \\
\therefore x &= \pm 4 \\
\text{But} \ x \ \text{must be negative (diagram)} \\
\therefore x &= -4
\end{align*}
\]

\[
\begin{align*}
\text{NB!! Just a reminder that:} \\
\sin(A - B) \neq \sin(A) - \sin(B)
\end{align*}
\]

\[
\begin{align*}
(1) \quad & \sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta \\
\therefore \sin(\alpha - \beta) &= \left(\frac{3}{5}\right) \left(\frac{12}{13}\right) - \left(\frac{-4}{5}\right) \left(\frac{-5}{13}\right) \\
\therefore \sin(\alpha - \beta) &= \frac{36}{65} - \frac{20}{65} = \frac{16}{65}
\end{align*}
\]

\[
\begin{align*}
(2) \quad & \cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta \\
\therefore \cos(\alpha + \beta) &= \left(\frac{-4}{5}\right) \left(\frac{12}{13}\right) - \left(\frac{3}{5}\right) \left(\frac{-5}{13}\right) \\
\therefore \cos(\alpha + \beta) &= \frac{-48}{65} + \frac{15}{65} = \frac{-33}{65}
\end{align*}
\]

\[
\begin{align*}
(3) \quad & \cos(90^\circ - \alpha + \beta) \\
&= \cos(90^\circ - (\alpha - \beta)) \\
&= \sin(\alpha - \beta) \\
&= \frac{16}{65} \quad \text{determined in (1)}
\end{align*}
\]

\[\beta\]

\[
\begin{align*}
x^2 + y^2 &= r^2 \quad \text{Pythagoras.} \\
x &= 12 \quad \text{and} \quad r = 13 \\
y^2 &= r^2 - x^2 \\
\therefore y^2 &= (13)^2 - (12)^2 \\
\therefore y^2 &= 25 \\
\therefore y &= \pm 5 \\
\text{But} \ y \ \text{must be negative (diagram)} \\
\therefore y &= -5
\end{align*}
\]

\[
\begin{align*}
&\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta \\
&\therefore \sin(\alpha - \beta) &= \left(\frac{3}{5}\right) \left(\frac{12}{13}\right) - \left(\frac{-4}{5}\right) \left(\frac{-5}{13}\right) \\
&\therefore \sin(\alpha - \beta) &= \frac{36}{65} - \frac{20}{65} = \frac{16}{65}
\end{align*}
\]

\[
\begin{align*}
&\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta \\
&\therefore \cos(\alpha + \beta) &= \left(\frac{-4}{5}\right) \left(\frac{12}{13}\right) - \left(\frac{3}{5}\right) \left(\frac{-5}{13}\right) \\
&\therefore \cos(\alpha + \beta) &= \frac{-48}{65} + \frac{15}{65} = \frac{-33}{65}
\end{align*}
\]

\[
\begin{align*}
&\cos(90^\circ - \alpha + \beta) \\
&= \cos(90^\circ - (\alpha - \beta)) \\
&= \sin(\alpha - \beta) \\
&= \frac{16}{65}
\end{align*}
\]

\[\text{EXAMPLE 11}\]

\[
\begin{align*}
(a) \quad & \text{If} \ \cos 25^\circ = k \ \text{determine the following in terms of} \ k. \\
&(1) \ \cos 155^\circ \quad (2) \ \cos 50^\circ \quad (3) \ \sin 25^\circ \quad (4) \ \sin 50^\circ \\
(b) \quad & \text{If} \ \sin 18^\circ = t \ \text{determine the following in terms of} \ t. \\
&(1) \ \cos 18^\circ \quad (2) \ \sin 78^\circ \quad (3) \ \sin 36^\circ \quad (4) \ \tan 198^\circ
\end{align*}
\]

\[
\begin{align*}
&\text{Solutions} \\
&\text{For the above examples it is advisable to draw a diagram.}
\end{align*}
\]
(a) \begin{align*}
\cos 25^\circ &= \frac{k}{1} \\
y^2 &= r^2 - x^2 \\
\therefore y^2 &= (1)^2 - (k)^2 \\
\therefore y^2 &= 1 - k^2 \\
\therefore y &= \sqrt{1 - k^2}
\end{align*}

(1) \cos 155^\circ = \cos(180^\circ - 25^\circ) \quad \text{(reduce to 155° to an acute angle)}
\therefore \cos 155^\circ = -\cos 25^\circ
\therefore \cos 155^\circ = -k

(2) \cos 50^\circ = \cos\left[2\left(25^\circ\right)\right] \quad \text{(write 50° as a double angle)}
\therefore \cos 50^\circ = 2\cos^2 25^\circ - 1
\therefore \cos 50^\circ = 2k^2 - 1

(3) \sin 25^\circ = \frac{y}{r} = \frac{\sqrt{1-k^2}}{1} = \sqrt{1-k^2}

(4) \sin 50^\circ = \sin\left[2\left(25^\circ\right)\right] \quad \text{(write 50° as a double angle)}
\therefore \sin 50^\circ = 2\sin 25^\circ \cdot \cos 25^\circ
\therefore \sin 50^\circ = 2\sqrt{1-k^2} \cdot k
\therefore \sin 50^\circ = 2k\sqrt{1-k^2}

(b) \begin{align*}
\sin 18^\circ &= \frac{t}{1} \\
x^2 &= r^2 - y^2 \\
\therefore x^2 &= (1)^2 - (t)^2 \\
\therefore x^2 &= 1 - t^2 \\
\therefore x &= \sqrt{1 - t^2}
\end{align*}

(1) \cos 18^\circ = \frac{x}{r} = \frac{\sqrt{1-t^2}}{1} = \sqrt{1-t^2}

(2) \sin 78^\circ = \sin\left(60^\circ + 18^\circ\right)
\therefore \sin 78^\circ = \sin 60^\circ \cdot \cos 18^\circ + \cos 60^\circ \cdot \sin 18^\circ
\therefore \sin 78^\circ = \left(\frac{\sqrt{3}}{2}\right) \sqrt{1-t^2} + \left(\frac{1}{2}\right) t
\therefore \sin 78^\circ = \frac{\sqrt{3} \sqrt{1-t^2}}{2} + \frac{t}{2}
\therefore \sin 78^\circ = \frac{\sqrt{3(1-t^2)} + t}{2}
(3) \( \sin 36^{\circ} = \sin \left[ 2 \left( 18^{\circ} \right) \right] \)  
\( \therefore \sin 36^{\circ} = 2 \sin 18^{\circ} \cos 18^{\circ} \)
\( \therefore \sin 36^{\circ} = 2(\sin 18^{\circ}) \sqrt{1-t^2} \)
\( \therefore \sin 36^{\circ} = 2 \sin 18^{\circ} \sqrt{1-t^2} \)

(4) \( \tan 198^{\circ} = \tan \left( 180^{\circ} + 18^{\circ} \right) \)
\( \therefore \tan 198^{\circ} = \tan 18^{\circ} \)
\( \therefore \tan 198^{\circ} = \frac{t}{\sqrt{1-t^2}} \)
using the diagram
or alternatively we can use the identity:
\( \tan 18^{\circ} = \frac{\sin 18^{\circ}}{\cos 18^{\circ}} = \frac{t}{\sqrt{1-t^2}} \)

**EXERCISE 5**

1. If \( \sin \theta = -\frac{5}{13} \) and \( \theta \in \left( 90^{\circ} ; 270^{\circ} \right) \) calculate without the use of a calculator and with the aid of a diagram the value of:
   (a) \( \sin 20^{\circ} \)  
   (b) \( \cos 20^{\circ} \)  

2. If \( 3 \cos \theta = \sqrt{5} \) and \( \theta \in \left( 180^{\circ} ; 360^{\circ} \right) \) calculate without the use of a calculator and with the aid of a diagram the value of:
   (a) \( \sin 20^{\circ} \)  
   (b) \( \cos 20^{\circ} \)  
   (c) \( \tan 20^{\circ} \)  

3. If \( \tan \alpha = \frac{12}{5} \) with \( \alpha \in \left( 90^{\circ} ; 270^{\circ} \right) \) and \( \cos \beta = -\frac{8}{17} \) with \( \beta \in \left( 0^{\circ} ; 180^{\circ} \right) \) calculate without the use of a calculator and with the aid of a diagram the value of \( \sin \left( \alpha + \beta \right) \).

4. If \( 2 \sin \alpha = -1 \) with \( \alpha \in \left( 90^{\circ} ; 270^{\circ} \right) \) and \( \tan \beta = \frac{2}{\sqrt{12}} \) with \( \beta \in \left( 90^{\circ} ; 270^{\circ} \right) \) calculate without the use of a calculator and with the aid of a diagram the value of:
   (a) \( \cos \left( \alpha - \beta \right) \)  
   (b) \( \cos 2 \alpha - \cos 2 \beta \)  

5. If \( \sin 2A = \frac{\sqrt{3}}{3} \), with \( 2A \in \left[ 90^{\circ} ; 270^{\circ} \right] \) determine without the use of a calculator the value of:
   (a) \( \cos 2A \)  
   (b) \( \tan 2A \)  
   (c) \( \sin A \)  

6. If \( \cos 36^{\circ} = k \) determine the following in terms of \( k \).
   (a) \( \cos(-36^{\circ}) \)  
   (b) \( \cos 72^{\circ} \)  
   (c) \( \sin 36^{\circ} \)  
   (d) \( \sin 72^{\circ} \)  
   (e) \( \tan 36^{\circ} \)  

7. If \( \sin 12^{\circ} = k \) determine the following in terms of \( k \).
   (a) \( \cos 24^{\circ} \)  
   (b) \( \sin 24^{\circ} \)  
   (c) \( \sin 78^{\circ} \)  
   (d) \( \sin 42^{\circ} \)  
   (e) \( \tan 12^{\circ} \)  

8. If \( \sin 9^{\circ} = k \) determine the value of \( \sin 18^{\circ} \) in terms of \( k \).

9. If \( \cos 55^{\circ} = p \) determine the value of \( \cos 5^{\circ} \) in terms of \( p \).
IDENTITIES INVOLVING COMPOUND AND DOUBLE ANGLES

An identity is a mathematical statement that is true for all values of the variable excluding the values the statement is not defined for. In Grade 12, compound and double angles are included in this topic.

Tools you can use to prove identities:

1. Split left-hand side (LHS) and right-hand side (RHS).
2. Use the identities $\tan \theta = \frac{\sin \theta}{\cos \theta}$ and $\sin^2 \theta + \cos^2 \theta = 1$.
3. Use double and compound angle formulae.
4. Addition and subtraction of fractions $\Rightarrow$ Find LCD and simplify.
5. Factorise (take out the HCF, difference of two squares, trinomials, grouping, sum or difference of two cubes).

EXAMPLE 12

Prove that: $\frac{1 - \cos 2\theta}{\sin 2\theta} = \tan \theta$

Solution

Firstly, it is important to split the identity so that you can work with the LHS and RHS separately.

LHS: $\frac{1 - \cos 2\theta}{\sin 2\theta}$

RHS: $\tan \theta = \frac{\sin \theta}{\cos \theta}$

The right-hand side has only one function value of $\theta$ while the left-hand side has more than one function value of $\theta$.

The LHS can now be simplified using double angle formulae.

$\therefore$ LHS $= \frac{1 - (1 - 2\sin^2 \theta)}{2 \sin \theta \cos \theta}$

or

$\therefore$ LHS $= \frac{1 - (\cos^2 \theta - \sin^2 \theta)}{2 \sin \theta \cos \theta}$

Note that any of the three double angle formulae for $\cos 2\theta$ could have been used. However, with practice you will be able to identify which of the three is the most appropriate to use.

$\therefore$ LHS $= \frac{1 - 1 + 2\sin^2 \theta}{2 \sin \theta \cos \theta}$

or

$\therefore$ LHS $= \frac{1 - \cos^2 \theta + \sin^2 \theta}{2 \sin \theta \cos \theta}$

$\therefore$ LHS $= \frac{2\sin^2 \theta}{2 \sin \theta \cos \theta}$

or

$\therefore$ LHS $= \frac{\sin^2 \theta + \sin^2 \theta}{2 \sin \theta \cos \theta}$

$\therefore$ LHS $= \frac{\sin \theta}{\cos \theta} = \text{RHS}$

or

$\therefore$ LHS $= \frac{\sin \theta}{\cos \theta} = \text{RHS}$
EXAMPLE 13

(a) Prove that: \[\frac{1 + \sin 2A}{\cos 2A} = \frac{\cos A + \sin A}{\cos A - \sin A}\]

Solution

Method 1

LHS: \[\frac{1 + \sin 2A}{\cos 2A}\]

RHS: \[\frac{\cos A + \sin A}{\cos A - \sin A}\]

For the LHS, the most appropriate identity to use for \(\cos 2A\) is \(\cos^2 A - \sin^2 A\) since the RHS contains the expression \(\cos A - \sin A\) in the denominator and we know that \(\cos^2 A - \sin^2 A = (\cos A - \sin A)(\cos A + \sin A)\)

\[\therefore LHS = \frac{1 + 2\sin A \cdot \cos A}{\cos^2 A - \sin^2 A}\]

\[\therefore LHS = \frac{\cos^2 A + \sin^2 A + 2\sin A \cdot \cos A}{\cos^2 A - \sin^2 A} \quad (1 = \cos^2 A + \sin^2 A)\]

\[\therefore LHS = \frac{\sin^2 A + 2\sin A \cdot \cos A + \cos^2 A}{\cos^2 A - \sin^2 A}\]

\[\therefore LHS = \frac{(\cos A + \sin A)(\cos A + \sin A)}{(\cos A - \sin A)(\cos A + \sin A)}\]

\[\therefore LHS = \frac{\cos A + \sin A}{\cos A - \sin A} = \text{RHS}\]

Method 2

LHS = \[\frac{1 + 2\sin A \cdot \cos A}{\cos^2 A - \sin^2 A}\]

and RHS = \[\frac{\cos A + \sin A}{\cos A - \sin A}\]

LHS = \[\frac{1 + 2\sin A \cdot \cos A}{(\cos A - \sin A)(\cos A + \sin A)}\]

Find the LCD for both LHS and RHS: \((\cos A - \sin A)(\cos A + \sin A)\)

\[\therefore \text{RHS} = \frac{\cos A + \sin A}{\cos A - \sin A} \times \frac{(\cos A + \sin A)}{(\cos A + \sin A)}\]

\[\therefore \text{RHS} = \frac{(\cos A + \sin A)^2}{(\cos A - \sin A)(\cos A + \sin A)}\]

\[\therefore \text{RHS} = \frac{\cos^2 A + 2\sin A \cdot \cos A + \sin^2 A}{(\cos A - \sin A)(\cos A + \sin A)}\]

Substitute \(\cos^2 A + \sin^2 A = 1\)

\[\therefore \text{RHS} = \frac{1 + 2\sin A \cdot \cos A}{(\cos A - \sin A)(\cos A + \sin A)} = \text{LHS}\]
(b) Hence, calculate \( \frac{\cos 15^\circ + \sin 15^\circ}{\cos 15^\circ - \sin 15^\circ} \) without the use of a calculator.

Using the identity \( \frac{1 + \sin 2A}{\cos 2A} = \frac{\cos A + \sin A}{\cos A - \sin A} \) we can let \( A = 15^\circ \):

\[
\begin{align*}
\frac{\cos 15^\circ + \sin 15^\circ}{\cos 15^\circ - \sin 15^\circ} &= \frac{1 + \sin 2(15^\circ)}{\cos 2(15^\circ)} = \frac{1 + \sin 30^\circ}{\cos 30^\circ} = \frac{1 + \left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{3}{\sqrt{3}} = \sqrt{3}
\end{align*}
\]

EXERCISE 6

1. Prove the following:

(a) \((\cos \theta + \sin \theta)^2 = 1 + \sin 2\theta\) (A useful result to remember)

(b) \(\frac{\cos 2x}{\cos x - \sin x} = \cos x + \sin x\)

(c) \(\cos^4 \alpha - \sin^4 \alpha = \cos 2\alpha\)

(d) \(\frac{1 - \sin 2x}{\sin x - \cos x} = \sin x - \cos x\)

(e) \(\frac{\sin \theta + \sin 2\theta}{1 + \cos \theta + \cos 2\theta} = \tan \theta\)

(f) \(\frac{1 + \cos 2A}{\cos 2A} = \frac{\tan 2A}{\tan A}\)

(g) \(\tan A + \frac{\cos A}{\sin A} = \frac{2}{\sin 2A}\)

(h) \(\left(\frac{\theta}{2} + \sin \frac{\theta}{2}\right)^2 = 1 + \sin \theta\)

(i) \(\frac{\sin 4\theta - \sin 2\theta}{\cos 4\theta + \cos 2\theta} = \tan \theta\)

(j) \(\sin 4\theta = 4\sin \theta\cos \theta - 8\sin^3 \theta\cos \theta\)

2. (a) Prove that: \(\frac{1 - \tan A}{1 + \tan A} = \cos 2A\) (b) Hence, calculate \(\frac{1 - \tan 22.5^\circ}{1 + \tan 22.5^\circ}\)

3. (a) Prove that: \(\frac{\sin 2\alpha}{1 + \cos 2\alpha} = \tan \alpha\) (b) Hence, calculate \(\tan 75^\circ\).

TRIGONOMETRIC EQUATIONS

EXAMPLE 14

(a) Solve for \(\theta\) if \(\cos \theta \cos 2^\circ + \sin \theta \sin 24^\circ = 0.715\) and \(\theta \in \left[-90^\circ; 90^\circ\right]\)

(b) (1) Show that if \(\cos(\theta + 30^\circ) = \frac{1}{2} \sin \theta\) then \(\tan \theta = \frac{\sqrt{3}}{2}\)

(2) Hence find the general solution for \(\cos(\theta + 30^\circ) = \frac{1}{2} \sin \theta\).

Solution

(a) \(\cos \theta \cos 2^\circ + \sin \theta \sin 24^\circ = 0.715\)

\(\therefore \cos (\theta - 24^\circ) = 0.715\)

Ref. angle = 44.35680084°

Quad 1: \(\theta - 24^\circ = 44.35,\ldots^\circ + k.360^\circ\) or \(\theta - 24^\circ = 360^\circ - 44.35,\ldots^\circ + k.360^\circ\)

\(\theta = 68.36^\circ + k.360^\circ\) or \(\theta = 339.64^\circ + k.360^\circ\) \(k \in \mathbb{Z}\)
EXAMPLE 15

Find the general solution of:

(a) \( \sin 2\theta + 2\sin \theta = 0 \)

(b) \( 2\sin^2 \theta + \sin \theta = 3 \)

(c) \( \cos 2\theta + \cos \theta = 0 \)

Solutions

(a) \( \sin 2\alpha + 2\sin \alpha = 0 \)

Firstly you want the equation to be written in such a way that you have function values of the same angle.

\[ \therefore 2\sin \alpha \cos \alpha + 2\sin \alpha = 0 \quad \text{(expand using the double angle formulae)} \]

\[ \therefore 2\sin \alpha (\cos \alpha + 1) = 0 \quad \text{(factorise)} \]

\[ \therefore 2\sin \alpha = 0 \quad \text{or} \quad \cos \alpha + 1 = 0 \quad \text{(zero factor law)} \]

\[ \therefore \sin \alpha = 0 \quad \text{or} \quad \cos \alpha = -1 \]

Now solve each one separately and then write the combined solution at the end.

For \( \sin \alpha = 0 \): Ref. angle = \(0^\circ\)

Quad 1:
\[ \alpha = 0^\circ + k.360^\circ \quad \text{or} \quad \alpha = 180^\circ - 0^\circ + k.360^\circ \]

\[ \therefore \alpha = k.360^\circ \quad \text{or} \quad \alpha = 180^\circ + k.360^\circ \quad (k \in \mathbb{Z}) \]

For \( \cos \alpha = -1 \): Ref. angle = \(0^\circ\)

Quad 2:
\[ \alpha = 180^\circ - 0^\circ + k.360^\circ \quad \text{or} \quad \alpha = 180^\circ + 0^\circ + k.360^\circ \]

\[ \therefore \alpha = 180^\circ + k.360^\circ \quad (k \in \mathbb{Z}) \]

The general solution for \( \sin 2\alpha + 2\sin \alpha = 0 \) is:
\[ \alpha = k.360^\circ \quad \text{or} \quad \alpha = 180^\circ + k.360^\circ \quad (k \in \mathbb{Z}) \]
(b) \[2 \sin^2 \alpha + \sin \alpha = 3\]
\[2 \sin^2 \alpha + \sin \alpha - 3 = 0\]  \hspace{1cm} \text{(standard form)}
\[
\therefore (2 \sin \alpha + 3)(\sin \alpha - 1) = 0 \quad \text{(factorise the trinomial)}
\]
\[
\therefore 2 \sin \alpha + 3 = 0 \quad \text{or} \quad \sin \alpha - 1 = 0 \quad \text{(zero factor law)}
\]
\[
\therefore 2 \sin \alpha = -3 \quad \text{or} \quad \sin \alpha = 1
\]
\[
\therefore \sin \alpha = \frac{-3}{2} \quad \text{or} \quad \sin \alpha = 1
\]
\textbf{For } \sin \alpha = -\frac{3}{2}: \quad \text{No solution, because } -1 \leq \sin \alpha \leq 1

\textbf{For } \sin \alpha = 1: \quad \text{Ref. angle } = 90^\circ
\]
\[
\text{Quad 1:} \quad \text{Quad 2:}
\]
\[
\alpha = 90^\circ + k \cdot 360^\circ \quad \text{or} \quad \alpha = 180^\circ - 90^\circ + k \cdot 360^\circ
\]
\[
\therefore \alpha = 90^\circ + k \cdot 360^\circ \quad (k \in \mathbb{Z})
\]
\[
(c) \quad \cos 2\alpha + \cos \alpha = 0
\]
Expand using the double angle formula for \(\cos 2\alpha\) because 2\(\alpha\) and \(\alpha\) are different angles.
\[
\therefore 2 \cos^2 \alpha - 1 + \cos \alpha = 0
\]
\[
\therefore 2 \cos^2 \alpha + \cos \alpha - 1 = 0
\]
\[
\therefore (2 \cos \alpha - 1)(\cos \alpha + 1) = 0 \quad \text{(factorise)}
\]
\[
\therefore 2 \cos \alpha - 1 = 0 \quad \text{or} \quad \cos \alpha + 1 = 0 \quad \text{(zero factor law)}
\]
\[
\therefore 2 \cos \alpha = 1 \quad \text{or} \quad \cos \alpha = -1
\]
\[
\therefore \cos \alpha = \frac{1}{2} \quad \text{or} \quad \cos \alpha = -1
\]
\textbf{For } \cos \alpha = -\frac{1}{2}: \quad \text{Ref. angle } = 60^\circ
\]
\[
\text{Quad 1:} \quad \text{Quad 4:}
\]
\[
\alpha = 60^\circ + k \cdot 360^\circ \quad \text{or} \quad \alpha = 360^\circ - 60^\circ + k \cdot 360^\circ
\]
\[
\therefore \alpha = 60^\circ + k \cdot 360^\circ \quad \text{or} \quad \alpha = 300^\circ + k \cdot 360^\circ \quad (k \in \mathbb{Z})
\]
\textbf{For } \cos \alpha = 1: \quad \text{Ref. angle } = 0^\circ
\]
\[
\text{Quad 2:} \quad \text{Quad 3:}
\]
\[
\alpha = 180^\circ - 0^\circ + k \cdot 360^\circ \quad \text{or} \quad \alpha = 180^\circ + 0^\circ + k \cdot 360^\circ
\]
\[
\therefore \alpha = 180^\circ + k \cdot 360^\circ \quad (k \in \mathbb{Z})
\]

\textbf{EXAMPLE 16}

Solve for \(\alpha\) if \(1 + \sin 2\alpha - 4 \sin^2 \alpha = 0\) and \(\alpha \in [-180^\circ ; 90^\circ]\)

\textbf{Solution}
\[
1 + \sin 2\alpha - 4 \sin^2 \alpha = 0
\]
\[
\therefore (\sin^2 \alpha + \cos^2 \alpha) + \sin 2\alpha - 4 \sin^2 \alpha = 0 \quad \text{[1 = (sin^2 \alpha + cos^2 \alpha)]}
\]
\[
\therefore \cos^2 \alpha + 2 \sin \alpha \cdot \cos \alpha - 3 \sin^2 \alpha = 0 \quad \text{(expand } \sin 2\alpha)\]
\[
\therefore (\cos \alpha - \sin \alpha)(\cos \alpha + 3 \sin \alpha) = 0 \quad \text{(factorise the trinomial)}
\]
\[
\therefore \cos \alpha - \sin \alpha = 0 \quad \text{or} \quad \cos \alpha + 3 \sin \alpha = 0 \quad \text{(zero factor law)}
\]
\[
\therefore -\sin \alpha = -\cos \alpha \quad \text{or} \quad 3 \sin \alpha = -\cos \alpha
\]
When solving a trigonometric equation of this form, you have to divide by \( \cos \alpha \) so that you can work with one trigonometric function only.

\[
\therefore \frac{\sin \alpha}{\cos \alpha} = \cos \alpha \quad \text{or} \quad \frac{3 \sin \alpha}{\cos \alpha} = -\cos \alpha
\]

\[
\therefore \frac{\cos \alpha}{\sin \alpha} = \cos \alpha \quad \text{or} \quad \frac{3 \sin \alpha}{\cos \alpha} = -\cos \alpha
\]

\[
\therefore \tan \alpha = 1 \quad \text{or} \quad 3 \tan \alpha = -1
\]

For \( \tan \alpha = 1 \): Ref angle = 45°

Quad 1: \( \alpha = 45° + k.180° \) or \( \alpha = 180° + 45° + k.180° \)

Quad 3: \( \alpha = 180° + 45° + k.180° \)

\[
\therefore \alpha = 45° + k.180° \quad \text{or} \quad \alpha = 225° + k.180°
\]

For \( \tan \alpha = -\frac{1}{3} \): Ref angle = 18.43...

Quad 2: \( \alpha = 180° - 18.43° + k.180° \) or \( \alpha = 360° - 18.43° + k.180° \)

Quad 4: \( \alpha = 180° + 360° - 18.43° + k.180° \)

\[
\therefore \alpha = 161.57° + k.180° \quad \text{or} \quad \alpha = 341.57° + k.180°
\]

\[
\begin{array}{|c|c|c|c|}
\hline
\alpha & k = -2 & k = -1 & k = 0 \\
\hline
45° + k.180° & -315° N/A & -135° & 45° \\
225° + k.180° & -135° & 45° & 225° N/A \\
161.57° + k.180° & -198.43° N/A & -18.43° & 161.57° N/A \\
341.57° + k.180° & -18.43° & 161.57° N/A & 341.57° N/A \\
\hline
\end{array}
\]

\( \alpha \in \{-135° ; -18.43° ; 45°\} \)

**EXERCISE 7**

1. Determine the general solution of:
   (a) \( \cos 40° \cos 40° + \sin 40° \sin 40° = -1 \)
   (b) \( \sin \theta = \sin 20° \)
   (c) \( 2 \sin^2 \theta - 3 \sin \theta + 1 = 0 \)
   (d) \( \cos 2\theta = \cos \theta + 1 \)
   (e) \( 3 \sin^2 \theta + 2 \sin 2\theta + \cos^2 \theta = 0 \)
   (f) \( 2 \cos \theta - \cos 2\theta = 1 \)
   (g) \( \sin^2 \theta - \sin 2\theta + \cos^2 \theta = 0 \)

2. (a) Show that \( \frac{\sin 2x}{\cos 2x - 1} = -\frac{\cos x}{\sin x} \)
   (b) Hence, solve for \( x \) if \( \frac{\sin 2x}{\cos 2x - 1} = 1 \) and \( x \in [0° ; 360°] \)

3. Solve for \( \beta \) if \( \beta \in (-180° ; 270°) \)
   (a) \( \sin 2\beta - \cos \beta = 0 \)
   (b) \( \cos 2\beta \sin \beta - \cos \beta \sin 2\beta = 0.5 \)
   (c) \( \tan \beta = \sin \beta \)

4. (a) Show that \( \sin (\theta - 30°) - \sin (\theta + 30°) = -\cos \theta \)
   (b) Determine the general solution for \( \sin (\theta - 30°) - \sin (\theta + 30°) = 0.5 \)

5. Solve for \( \theta \) if \( \sin(\theta + 60°) = 2 \sin \theta \) where \( \theta \in (-180° ; 90°) \)

6. Determine the general solution of \( 4 \cos^2(-x) - \sin(2x - 180°) = \tan^2 675° \)
   where \( \tan x > 0 \).
In some trigonometric equations, the right side of the equation can be expressed using an appropriate reduction formula. The angles are then equated thus yielding two general solutions. There are six types of equations that can be considered.

**EXAMPLE 17**

Determine the general solution of the following equations:

(a) \( \cos 3x = \sin x \)  \hspace{1cm} (b) \( \sin (3\theta + 30^\circ) = -\cos 2\theta \)

**Solutions**

(a) In this example, we will start by making use of the following reduction rule on the right side of the equation: \( \sin x = \cos(90^\circ - x) \)

Using this rule, the right side of the equation \( \cos 3x = \sin x \) can also be expressed in terms of the cosine of an angle as follows:

\[ \cos 3x = \cos(90^\circ - x) \]

which is of the form: \( \cos A = \cos B \)

Two cases to consider are: \( \cos A = \cos B \) or \( \cos A = \cos(360^\circ - B) \)

**Case 1**

One general solution of the equation can be obtained by equating the angles and adding \( k.360^\circ \):

\[ \cos 3x = \cos(90^\circ - x) \]

\[ \therefore 3x = (90^\circ - x) + k.360^\circ \]

\[ \therefore 4x = 90^\circ + k.360^\circ \]

\[ \therefore x = 22.5^\circ + k.90^\circ \]

**Case 2**

There is another general solution. If we use the reduction rule \( \cos \theta = \cos(360^\circ - \theta) \), the equation can be rewritten as follows:

\[ \cos 3x = \cos(360^\circ - 90^\circ) \]

\[ \therefore 3x = 360^\circ - (90^\circ - x) + k.360^\circ \]

\[ \therefore 3x = 270^\circ + x + k.360^\circ \]

\[ \therefore x = 135^\circ + k.180^\circ \]
(b) \[ \sin(30 + 30°) = -\cos 20 \]
\[ \therefore \sin(30 + 30°) = -\sin(90° - 20) \quad [\cos x = \sin(90° - x)] \]

**Case 1**

\[ \sin(30 + 30°) = -\sin(90° - 20) \]
\[ \therefore \sin(30 + 30°) = \sin(180° + (90° - 20)) \quad [\sin A = \sin(180° + B)] \]
\[ \therefore \sin(30 + 30°) = \sin(270° - 20) \]
\[ \therefore 30 + 30° = 270° - 20 + k \cdot 360° \quad \text{where } k \in \mathbb{Z} \]
\[ \therefore 50 = 240° + k \cdot 360° \]
\[ \therefore \theta = 48° + k \cdot 72° \]

**Case 2**

\[ \sin(30 + 30°) = -\sin(90° - 20) \]
\[ \therefore \sin(30 + 30°) = \sin(360° - (90° - 20)) \quad [\sin A = \sin(360° - B)] \]
\[ \therefore \sin(30 + 30°) = \sin(270° + 20) \]
\[ \therefore 30 + 30° = 270° + 20 + k \cdot 360° \quad \text{where } k \in \mathbb{Z} \]
\[ \therefore \theta = 240° + k \cdot 360° \]

**EXERCISE 8**

Determine the general solution of:
(a) \[ \cos(20° + 45°) = \cos(20° - \theta) \]
(b) \[ \sin 3\theta = \sin(20 + 10°) \]
(c) \[ \cos(\theta + 30°) = \sin 2\theta \]
(d) \[ \sin(\theta - 15°) = \cos(20° + \theta) \]
(e) \[ \sin(\theta - 60°) = -\cos 20 \]
(f) \[ \tan 20 = -\tan \theta \]

**THE VALUES FOR WHICH IDENTITIES ARE UNDEFINED**

Remember that an identity is a statement of equality that is true for all values of the variable (except for those values the identity is not defined for). You will now be required to determine values of the variable for which an identity is undefined. Consider the following:

- An identity is undefined for values for which the denominator is zero.
- The tangent function is undefined at its asymptotes. The equations of the asymptotes for any tangent function of the form \( f(\theta) = \tan \theta \) are \( \theta = 90° + k \cdot 180° \) \((k \in \mathbb{Z})\).

Since \( \tan \theta = \frac{\sin \theta}{\cos \theta} \), it is important to note that \( \tan \theta \) will also be undefined for \( \cos \theta = 0 \) (denominator may not be equal to 0).

Note: The general solution of the equation \( \cos \theta = 0 \) is:
\[ x = 90° + k \cdot 360° \quad \text{or} \quad x = 270° + k \cdot 360° \quad \text{where } k \in \mathbb{Z} \]

These general solutions can be written as \( x = 90° + k \cdot 180° \). Why is this so?

**EXAMPLE 18**

(a) For which values of \( \theta \) is the identity \( \frac{1 - \cos 2\theta}{\sin 2\theta} = \tan \theta \) undefined?
This identity involves the tangent function and therefore this identity is undefined for: \( \theta = 90^\circ + k.180^\circ \) \((k \in \mathbb{Z})\)

This identity is also undefined for those values of \( \theta \) for which the denominator is zero.

\[
\therefore \sin 2\theta = 0
\]

\[
\therefore 2\theta = 0^\circ + k.360^\circ \quad \text{or} \quad 2\theta = 180^\circ - 0^\circ + k.360^\circ
\]

\[
\therefore \theta = 0^\circ + k.180^\circ \quad \text{or} \quad \theta = 90^\circ + k.180^\circ
\]

Final solution: \( \theta = 0^\circ + k.180^\circ \) \((k \in \mathbb{Z})\) or \( \theta = 90^\circ + k.180^\circ \) \((k \in \mathbb{Z})\)

(b) For which values of \( A \) is \( \frac{1 - \tan A}{1 + \tan A} = \frac{\cos 2A}{1 + \sin 2A} \) undefined?

This identity involves the tangent function which is undefined for:

\( A = 90^\circ + k.180^\circ \) \((k \in \mathbb{Z})\)

This identity is also undefined for those values of \( A \) for which the denominator is zero.

\[
\therefore 1 + \tan A = 0 \quad \text{or} \quad 1 + \sin 2A = 0
\]

\[
\therefore \tan A = -1 \quad \text{or} \quad \sin 2A = -1
\]

For \( \tan A = -1 \):

Ref. angle = 45°

\[
\text{Quad 2:} \quad A = 180^\circ - 45^\circ + k.180^\circ
\]

\[
\text{or} \quad A = 135^\circ + k.180^\circ \quad (k \in \mathbb{Z})
\]

\[
\therefore 2A = 180^\circ + 90^\circ + k.360^\circ
\]

\[
\therefore 2A = 270^\circ + k.360^\circ
\]

\[
\therefore A = 135^\circ + k.180^\circ \quad (k \in \mathbb{Z})
\]

Ref. angle = 90°

\[
\text{Quad 3:} \quad \therefore A = 360^\circ - 45^\circ + k.180^\circ
\]

\[
\text{or} \quad A = 315^\circ + k.180^\circ \quad (k \in \mathbb{Z})
\]

\[
\therefore 2A = 360^\circ - 90^\circ + k.360^\circ
\]

\[
\therefore 2A = 270^\circ + k.360^\circ
\]

\[
\therefore A = 135^\circ + k.180^\circ \quad (k \in \mathbb{Z})
\]

Final solution:

\( A = 90^\circ + k.180^\circ \) or \( A = 135^\circ + k.180^\circ \) or \( A = 315^\circ + k.180^\circ \) \((k \in \mathbb{Z})\)

**EXERCISE 9**

The identities below were proved in Exercise 6 without stating any restrictions. Determine the values of the variable for which each of the following identities are undefined.

(a) \( \frac{\cos 2x}{\cos x - \sin x} = \cos x + \sin x \)  
(b) \( \frac{1 - \sin 2x}{\sin x - \cos x} = \sin x - \cos x \)

(c) \( \frac{\sin \theta + \sin 2\theta}{1 + \cos \theta + \cos 2\theta} = \tan \theta \)  
(d) \( \frac{1 + \cos 2A}{\tan 2A} = \frac{\tan 2A}{\cos 2A} \)

(e) \( \frac{\tan A + \frac{\cos A}{\sin A}}{\frac{2}{\sin 2A}} = \tan \alpha \)  
(f) \( \frac{\sin 4\theta - \sin 2\theta}{\cos 4\theta + \cos 2\theta} = \tan \theta \)

(g) \( \frac{\sin 2\alpha}{1 + \cos 2\alpha} = \tan \alpha \)

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TRIGONOMETRIC GRAPHS

In Grade 10 and 11 you studied amplitude shifts, vertical shifts, period shifts and horizontal shifts. The graphs were of the form:

\[ y = a \sin b(x + p) + q \], \[ y = a \cos b(x + p) + q \] and \[ y = a \tan b(x + p) + q \].

In the following exercise, you will revise these concepts.

EXERCISE 10

1. Sketch the graphs of the following. Write down the maximum value, minimum value, amplitude and period.
   (a) \[ y = 2 \sin x - 1 \] for \( x \in [0^\circ; 360^\circ] \)
   (b) \[ y = 1 - 2 \cos x \] for \( x \in [0^\circ; 360^\circ] \)
   (c) \[ y = \cos 2x \] for \( x \in [-45^\circ; 315^\circ] \)
   (d) \[ y = \cos \frac{1}{2}x \] for \( x \in [-360^\circ; 720^\circ] \)
   (e) \[ y = -2 \sin 2x \] for \( x \in [-90^\circ; 180^\circ] \)
   (f) \[ y = \frac{1}{2} \cos 2x \] for \( x \in [-90^\circ; 180^\circ] \)
   (g) \[ y = \cos(x + 60^\circ) \] for \( x \in [-150^\circ; 300^\circ] \)
   (h) \[ y = \sin(x - 45^\circ) \] for \( x \in [-135^\circ; 405^\circ] \)
   (i) \[ y = 2 \tan x \] for \( x \in [0^\circ; 360^\circ] \)

2. In each case, sketch the graphs on the same set of axes:
   (a) \[ y = -2 \sin x \] and \[ y = -2 + \sin x \] for \( x \in [-90^\circ; 180^\circ] \)
   (b) \[ y = \cos 3x \] and \[ y = 3 \cos x \] for \( x \in [0^\circ; 360^\circ] \)
   (c) \[ y = -2 \sin(x - 30^\circ) \] and \[ y = \frac{1}{2} \cos(x + 30^\circ) \] for \( x \in [-30^\circ; 390^\circ] \)
   (d) \[ y = \tan x - 1 \] and \[ y = 2 \tan x \] for \( x \in (-90^\circ; 180^\circ] \)

EXAMPLE 19

The diagram below represents the graphs of:

\[ f(x) = \tan x \] and \[ g(x) = \sin 2x \] for \( x \in [-90^\circ; 180^\circ] \)

(a) Determine the general solution of the equation \( \tan x = \sin 2x \).
(b) Solve this equation for \( x \in [-90^\circ; 180^\circ] \).
(c) What do these solutions represent?
Solutions

(a) \( \tan x = \sin 2x \)
    \[ \therefore \frac{\sin x}{\cos x} = 2 \sin x \cos x \]
    \[ \therefore \sin x = 2 \sin x \cos^2 x \]
    \[ \therefore \sin x - 2 \sin x \cos^2 x = 0 \]
    \[ \therefore \sin x(1 - 2 \cos^2 x) = 0 \]
    \[ \therefore \sin x = 0 \quad \text{or} \quad 1 - 2 \cos^2 x = 0 \]
    \[ 2 \cos^2 x - 1 = 0 \]
    \[ \cos 2x = 0 \]
    \[ x = 0^\circ + k \cdot 360^\circ \quad 2x = 90^\circ + k \cdot 360^\circ \]
    or \[ x = 180^\circ + k \cdot 360^\circ \quad \text{or} \]
    \[ x = 180^\circ + k \cdot 360^\circ \]
    where \( k \in \mathbb{Z} \)
    \[ 2x = 270^\circ + k \cdot 360^\circ \]
    \[ x = 135^\circ + k \cdot 180^\circ \]

<table>
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<th>( k )</th>
<th>( 0^\circ + k \cdot 360^\circ )</th>
<th>( 180^\circ + k \cdot 360^\circ )</th>
<th>( 45^\circ + k \cdot 180^\circ )</th>
<th>( 135^\circ + k \cdot 180^\circ )</th>
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<tr>
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<td>( 360^\circ )</td>
<td>( 540^\circ )</td>
<td>( 225^\circ )</td>
<td>( 315^\circ )</td>
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</table>

(b) For the interval \( x \in [-90^\circ ; 180^\circ] \): \( x \in \{-45^\circ ; 0^\circ ; 45^\circ ; 135^\circ ; 180^\circ \} \)

(c) The two graphs intersect at these values of \( x \).

EXAMPLE 20

The diagram below represents the graphs of:
\( f(x) = \tan x - 1 \) and \( g(x) = \cos 2x \) for \( x \in [-90^\circ ; 180^\circ] \)

(a) Determine graphically the value(s) of \( x \) for which:
1. \( \tan x = \cos 2x + 1 \)
2. \( 2 \cos^2 x - 1 = 0 \)
3. \( f(x) \leq 0 \)
4. \( g(x) > 0 \)
5. \( f(x) \geq g(x) \)
6. \( f(x) < g(x) \)
7. \( f(x) \equiv -1 \)
8. \( f(x).g(x) \geq 0 \)
9. \( f(x).g(x) < 0 \)
10. \( g(x) - f(x) = 2 \)
(b) For which values of $x$ is:

1. The graph of $f$ increasing?
2. The graph of $g$ decreasing?
3. $f'(x) > 0$
4. $g'(x) < 0$

**Solutions**

(a)(1) $\tan x = \cos 2x + 1$

\[ \therefore \tan x - 1 = \cos 2x \]

\[ \therefore f(x) = g(x) \]

\[ \therefore x = 45^\circ \]

(2) $2\cos^2 x - 1 = 0$

\[ \therefore \cos 2x = 0 \]

\[ \therefore x = -45^\circ \text{ or } x = 45^\circ \text{ or } x = 135^\circ \]

(3) $f(x) \leq 0$ (graph below $x$-axis or cutting it)

$-90^\circ < x \leq 45^\circ$ or $90^\circ < x \leq 180^\circ$

Alternative notation: $x \in (-90^\circ, 45^\circ] \cup (90^\circ, 180^\circ]$

(4) $g(x) > 0$ (graph above $x$-axis)

$-45^\circ < x < 45^\circ$ or $135^\circ < x \leq 180^\circ$

Alternative notation: $x \in (-45^\circ, 45^\circ) \cup (135^\circ, 180^\circ]$

The reason for including $180^\circ$ is because the graph is above the $x$-axis at this value.

(5) $f(x) \geq g(x)$ (f is above $g$ or intersecting)

$45^\circ \leq x < 90^\circ$

Alternative notation: $x \in [45^\circ, 90^\circ)$

(6) $f(x) < g(x)$ (f is below $g$)

$-90^\circ < x < 45^\circ$ or $90^\circ < x \leq 180^\circ$

Alternative notation: $x \in (-90^\circ, 45^\circ) \cup (90^\circ, 180^\circ]$

The reason for including $180^\circ$ is because the graph of $f$ is below the graph of $g$ at this value.

(7) $f(x) \geq -1$ (y-values of $f$ above or equal to $-1$)

$0^\circ \leq x < 90^\circ$ or $x = 180^\circ$

Alternative notation: $x \in [0^\circ, 90^\circ) \cup \{180^\circ\}$

(8) $f(x),g(x) \geq 0$ (product of y-values is greater than or equal to 0)

$-90^\circ < x \leq -45^\circ$ or $90^\circ < x \leq 135^\circ$ or $x = 45^\circ$
(9) \( f(x), g(x) < 0 \quad \text{(product of y-values is negative)} \)
\(-45^\circ < x < 45^\circ \) or \( 45^\circ < x < 90^\circ \) or \( 135^\circ < x \leq 180^\circ \)

Alternative notation:
\( x \in (-40^\circ ; 45^\circ) \cup (45^\circ ; 90^\circ) \cup (135^\circ ; 180^\circ) \]

(10) \( g(x) - f(x) = 2 \)
This is the value of \( x \) for which the y-value of \( g \) minus the y-value of \( f \) equals 2. \( x \in \{-45^\circ ; 0^\circ ; 135^\circ ; 180^\circ \} \)

(b)(1) The graph of \( f \) increases for all values of \( x \) where \( x \neq 90^\circ \)
(2) The graph of \( g \) decreases for \( 0^\circ < x < 90^\circ \)
(3) \( f'(x) > 0 \) means the values of \( x \) for which the gradient is positive or
where the graph is increasing. The solution is therefore all values of \( x \)
\( x \neq 90^\circ \)
(4) \( g'(x) < 0 \) means the values of \( x \) for which the gradient is negative or
where the graph is decreasing. The solution is therefore \( 0^\circ < x < 90^\circ \).

**EXERCISE 11**

1. Consider the functions: \( f(x) = \cos 2x \) and \( g(x) = -\sin x \)
   (a) Write down the period of \( f \).
   (b) Determine the values of \( x \in [-90^\circ ; 90^\circ] \) for which \( f(x) = g(x) \).
   (c) Sketch \( f \) and \( g \) on the same set of axes for \( x \in [-90^\circ ; 90^\circ] \).
   (d) Determine the values of \( x \) for which:
      (1) \( \cos 2x > -\sin x \)
      (2) \( \cos 2x < -\sin x \)
      (3) \( f(x) < 0 \)
      (4) \( g(x) \leq 0 \)
      (5) \( \sin x < 0 \)
      (6) \( f(x).g(x) < 0 \)
      (7) \( f(x).g(x) \geq 0 \)
      (8) \( g(x) - f(x) = 2 \)

2. (a) Solve for \( x \) if \( \cos(x - 30^\circ) = \sin 3x \) where \( x \in [-60^\circ ; 120^\circ] \).
   (b) Sketch the graphs of the following functions on the same set of axes for the interval \( x \in [-60^\circ ; 120^\circ] \):
      \( f(x) = \cos(x - 30^\circ) \) and \( g(x) = \sin 3x \)
   (c) Explain graphically what the solutions to the equation
      \( \cos(x - 30^\circ) = \sin 3x \) represent.
   (d) Determine graphically the values of \( x \) for which
      \( \cos(x - 30^\circ) > \sin 3x \)
   (e) For which values of \( x \) is \( f'(x) < 0 \) and \( g'(x) > 0 \)?

3. (a) Solve for \( x \) if \( -\tan x = \sin 2x \) where \( x \in [-90^\circ ; 90^\circ] \)
   (b) Draw neat sketch graphs of the following functions for the interval
      \( x \in [-90^\circ ; 90^\circ] \): \( f(x) = -\tan x \) and \( g(x) = \sin 2x \)
   (c) Hence determine graphically the values of \( x \) for which:
      \( \sin 2x \geq -\tan x \)
4. (a) Show that the equation \( 2 \cos \theta = \sin(\theta + 30^\circ) \) can be written as:
\[
\sqrt{3} \sin \theta = 3 \cos \theta
\]
(b) Hence solve the equation \( 2 \cos \theta = \sin(\theta + 30^\circ) \) for \( \theta \in [-180^\circ;180^\circ] \)
(c) On the same set of axes, draw the graphs of the following:
\[
f(\theta) = 2 \cos \theta \quad \text{and} \quad g(\theta) = \sin(\theta + 30^\circ) \quad \text{for} \quad \theta \in [-180^\circ;180^\circ]
\]
(d) Determine graphically the value(s) of \( \theta \) for which:
\[
\begin{align*}
(1) & \quad 2 \cos \theta > \sin(\theta + 30^\circ) \\
(2) & \quad \frac{2 \cos \theta}{\sin(\theta + 30^\circ)} \geq 0 \quad \text{where} \quad \theta \in [-180^\circ;0^\circ]
\end{align*}
\]
5. (a) Sketch the graphs of the following on the same set of axes for the interval \( x \in [-90^\circ;90^\circ] \):
\[
f(x) = 2 \cos x \quad \text{and} \quad g(x) = \tan 2x
\]
(b) Solve the equation \( 2 \cos x = \tan 2x \) for \( x \in [-90^\circ;90^\circ] \)
(c) Determine graphically the values of \( x \) for which \( f(x), g(x) \geq 0 \)
(d) Write down the period of \( f \left( \frac{x}{2} \right) \)
(e) Write down the asymptotes of \( g(x - 25^\circ) \)

**SOLUTION OF TRIANGLES IN THREE DIMENSIONS**

Whereas two-dimensional space occupies a single plane, three-dimensional space occupies three planes. The three planes are horizontal, vertical and inclined. The sine, cosine and area rules can also be used to solve problems in three dimensional space. The diagram below illustrates the three different planes for an object in three dimensions.
EXAMPLE 21

In the diagram, AD represents a flag pole of length 5 metres which is perpendicular to the horizontal plane. An observer at C notes that the angle of elevation of D is $35^\circ$, while another observer at B, in the same horizontal plane as C, finds that the angle of elevation of D is $42^\circ$. $\hat{BAC} = 65^\circ$

(a) What are the angles $\hat{DAC}$ and $\hat{DAB}$ equal to?
(b) How far is each observer from the foot of the flag pole? Round off to two decimal places.
(c) Calculate the distance between the two observers at C and B (two decimal places).
(d) Calculate the area of $\triangle ABC$ to the nearest whole number.

Solutions

Three-dimensional problems are solved in the same way that two-dimensional problems are. Identify what information is given with regard to sides and angles and then use the appropriate rule to calculate the lengths of sides and angles.

(a) $\hat{DAC} = 90^\circ$ and $\hat{DAB} = 90^\circ$. It is important to note that the flag pole is perpendicular to the horizontal plane and therefore it will be perpendicular to any point that lies on that plane.

(b) The length of AC and AB needs to be calculated. In $\triangle DAC$ and $\triangle DAB$, more than one angle and one side is given. The sine rule can therefore be used to calculate the length of AC and AB. Using the sum of the angles of a triangle, it can be deduced that:

$A\hat{DC} = 55^\circ$ and $A\hat{DB} = 48^\circ$.

In $\triangle DAC$:

$$\frac{\sin A}{a} = \frac{\sin D}{d} = \frac{\sin C}{c}$$

$$\therefore \frac{\sin 90^\circ}{\sin 55^\circ} = \frac{\sin 35^\circ}{5}$$

$$\therefore \frac{\sin 55^\circ}{\sin 35^\circ} = \frac{5}{AC}$$

$$\therefore 5 \sin 55^\circ = AC \sin 35^\circ$$

$$\therefore AC = \frac{5 \sin 55^\circ}{\sin 35^\circ}$$

$$\therefore AC = 7,14 \text{ m}$$

In $\triangle DAB$:

$$\frac{\sin A}{a} = \frac{\sin D}{d} = \frac{\sin B}{b}$$

$$\therefore \frac{\sin 90^\circ}{\sin 48^\circ} = \frac{\sin 42^\circ}{5}$$

$$\therefore \frac{\sin 48^\circ}{\sin 42^\circ} = \frac{5}{AB}$$

$$\therefore 5 \sin 48^\circ = AB \sin 42^\circ$$

$$\therefore AB = \frac{5 \sin 48^\circ}{\sin 42^\circ}$$

$$\therefore AB = 5,55 \text{ m}$$

Fill in your answers on the sketch.
In \( \triangle ABC \) there are two sides and an included angle. The cosine rule will be the appropriate rule to use to calculate the length of BC.

\[ BC = a, \ AC = b \text{ and } AB = c. \]

\[ \therefore a^2 = b^2 + c^2 - 2bc \cos A \]

\[ \therefore BC^2 = (7,14)^2 + (5,55)^2 - 2(7,14)(5,55) \cdot \cos 65^\circ \]

\[ \therefore BC^2 = 48,28791228 \]

\[ \therefore BC = 6,95 \text{ m} \]

Area \( \triangle ABC = \frac{1}{2} (7,14)(5,55) \sin 65^\circ = 18 \text{ m}^2 \)

**EXERCISE 12**

1. The figure shows an open birthday card which is 24,5cm long and 18,5cm wide. The card is opened at an angle of 105°.

Calculate:

(a) the length of PR (one decimal place)

(b) the length of RT (two decimal places).

(c) \( \hat{RPT} \) (two decimal places).

2. A and B lie in the same horizontal plane as C, the foot of a building DC. AB = 250 m, BC = 300 m and the angle of elevation of D from A is 28°. AB is perpendicular to BC.

(a) Calculate the height of the building (two decimal places).

(b) What is the magnitude of \( \hat{DCB} \)?

(c) Calculate \( \hat{ADB} \) (two decimal places).

3. A rectangular roof 12 m by 9 m forms an angle of 54° with the horizontal plane.

(a) What is the length of the pole AC?

(b) At what inclination will AC lie? (two decimal places)

(c) Determine \( \hat{DAC} \) (two decimal places)
4. In the diagram below, AB is a straight line 1 500 m long. DC is a vertical tower 158 m high with C, A and B points in the same horizontal plane. The angles of elevation of D from A and B are 25° and θ. CAB = 30°.
   (a) Determine the length of AC (two decimal places)
   (b) Find the value of θ (two decimal places)

5. A rectangular block of wood has a width of 6 metres, height of 8 metres and a length of 15 metres. A plane cut is made through the block as shown in the diagram revealing the triangular plane that has been formed.
   (a) Calculate the magnitude (size) of \( \hat{EBG} \).
   (b) Calculate the area of \( \triangle EBG \).

6. Two buildings GH and IJ are equal in height. K is a point in the same horizontal plane as G and J and \( GK = KJ = 20 \) m. The angle of elevation of both H and G from K is 32°. \( \hat{GKJ} = 110° \).
   Calculate:
   (a) the height of the buildings (one decimal place).
   (b) the distance between the buildings (two decimal places).

7. In the rectangular prism shown, determine:
   (a) the length of HF.
   (b) the inclination of CF (\( \hat{HFC} \)).
   (two decimal places)
8. A paperweight has the shape of a pyramid on a square base. Each of the faces is an equilateral triangle with sides 70 mm.
(a) What is the height of the top of the paper weight above its base?
(b) What is the angle between the base and one of the edges?

EXAMPLE 22

In the diagram, PQ is perpendicular to the horizontal plane RQS. QS = x metres; QRS = α; QSR = β, and the angle of elevation of P from R is θ.
(a) Determine the length of RQ in terms of α, β and x.
(b) Hence show that: \( PQ = \frac{x \sin \beta \tan \theta}{\sin \alpha} \)

Solutions

(a) RQ is common to \( \triangle PRQ \) and \( \triangle RQS \). We will first work with \( \triangle RQS \). The reason for this is that \( \triangle RQS \) has the most information and secondly and more importantly the question specifically stated that RQ must be in terms of α, β and x, which are all angles and sides of \( \triangle RQS \).

In \( \triangle RQS \):
\[
\frac{\sin Q}{q} = \frac{\sin R}{r} = \frac{\sin S}{s} \quad \text{(more than one angle and 1 side is given)}
\]
\[
\therefore \frac{\sin (180^\circ - (\alpha + \beta))}{RS} = \frac{\sin \alpha}{x} = \frac{\sin \beta}{RQ}
\]
\[
\therefore \frac{\sin \alpha}{x} = \frac{\sin \beta}{RQ} \quad \text{(choose the two that have the most information and contain the side we are looking for)}
\]
\[
\therefore RQ = \frac{x \sin \beta}{\sin \alpha}
\]

(b) PQ is in \( \triangle PRQ \) and \( \hat{P} = 90^\circ - \theta \) angles of a triangle
\[
\frac{\sin P}{p} = \frac{\sin Q}{q} = \frac{\sin R}{r} \quad \text{(more than one angle and one side is given)}
\]
\[
\therefore \frac{\sin (90^\circ - \theta)}{RQ} = \frac{\sin 90^\circ}{PR} = \frac{\sin \theta}{PQ}
\]
\[
\begin{align*}
\sin(90^\circ - \theta) &= \frac{\sin \theta}{RQ} \\
\therefore \cos \theta &= \frac{\sin \theta}{RQ} \\
\therefore PQL\cos \theta &= RQ \sin \theta \\
\therefore PQL &= \frac{RQ \sin \theta}{\cos \theta} \\
\therefore PQL &= RQ \tan \theta \\
\therefore PQL &= \left(\frac{x \sin \beta}{\sin \alpha}\right) \tan \theta \\
\therefore PQL &= \frac{x \sin \beta \tan \theta}{\sin \alpha}
\end{align*}
\]

**EXAMPLE 23**

In the figure alongside A, B and C are three points in the same horizontal plane. AD represents a lamp pole that is perpendicular to the horizontal plane.

Given: \(\hat{BDA} = \hat{ABC} = \theta\) and \(\hat{BCA} = \beta\). \(BC = x\).

(a) Write \(\hat{BAC}\) in terms of \(\theta\) and \(\beta\).

(b) Show that:
\[
\sin \theta = \sin \beta = \sin \left(\theta + \beta\right)
\]

(c) If \(AB = AC\) show that:
\[
\begin{align*}
(1) \quad AB &= \frac{x \sin \beta}{\sin \left(\theta + \beta\right)} \\
(2) \quad AD &= \frac{x}{2 \sin \theta}
\end{align*}
\]

**Solutions**

(a) \(\hat{A} = 180^\circ - (\theta + \beta)\) (angles of a triangle)

(b) \(\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}\) (more than one angle and one side is given)

\[
\begin{align*}
\therefore \frac{\sin \left(180^\circ - (\theta + \beta)\right)}{x} &= \frac{\sin \theta}{AC} = \frac{\sin \beta}{AB} \\
\therefore \frac{\sin \left(180^\circ - (\theta + \beta)\right)}{x} &= \frac{\sin \beta}{AB} \\
\therefore \frac{\sin (\theta + \beta)}{x} &= \frac{\sin \beta}{AB} \\
\therefore \sin (\theta + \beta) &= \frac{x \sin \beta}{AB} \\
\therefore AB \sin (\theta + \beta) &= x \sin \beta \\
\therefore AB &= \frac{x \sin \beta}{\sin (\theta + \beta)}
\end{align*}
\]
(c) (1) If \( AB = AC \) then \( \hat{B} = \hat{C} \) (isosceles triangle)

\[
\therefore \beta = \theta.
\]
\[
\therefore AB = \frac{x \sin \beta}{\sin(\theta + \beta)}
\]
\[
\therefore AB = \frac{x \sin \theta}{\sin(\theta + \theta)} \quad (\beta = 0)
\]
\[
\therefore AB = \frac{x \sin \theta}{\sin 2\theta}
\]
\[
\therefore AB = \frac{x \sin \theta}{2 \sin \theta \cos \theta}
\]
\[
\therefore AB = \frac{x}{2 \cos \theta}
\]

(2) In \( \triangle ABD \):

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin D}{d}
\]

(more than one angle and one side is given)

\[
\frac{\sin 90^\circ}{DB} = \frac{\sin(90^\circ - \theta)}{AD} = \frac{\sin \theta}{AB}
\]
\[
\therefore \frac{\sin (90^\circ - \theta)}{\sin \theta} = \frac{AB}{AD}
\]
\[
\therefore \frac{\cos \theta}{\sin \theta} = \frac{AB}{AD}
\]
\[
\therefore AB \cos \theta = AD \sin \theta
\]
\[
\therefore \frac{AB \cos \theta}{\sin \theta} = AD \quad \text{(substitute } AB = \frac{x}{2 \cos \theta})
\]
\[
\therefore \frac{\left(\frac{x}{2 \cos \theta}\right) \cos \theta}{\sin \theta} = AD
\]
\[
\therefore AD = \frac{x}{2 \sin \theta}
\]

**EXERCISE 13**

1. In the figure alongside Q, R and S are three points in the same horizontal plane. PS represents a flagpole that is perpendicular to the horizontal plane.

(a) Determine QS in terms of \( \alpha, \beta \) and \( x \)

(b) Hence, determine PS in terms of \( \alpha, \beta, \theta \) and \( x \)
2. In the following sketch, \( \triangle ABCD \) is in a horizontal plane. A is directly above B (AB is a vertical line). 
\( AB = h \), \( BC = CD = x \), \( \angle BAC = \alpha \) and \( \angle BCD = \theta \).
(a) Show that \( BD^2 = 2x^2(1 - \cos \theta) \)
(b) Hence show that \( h = \frac{x \sqrt{2(1 - \cos \theta)}}{\tan \alpha} \)
(c) Calculate the value of \( h \) if \( x = 100 \), \( \theta = 60^\circ \) and \( \alpha = 40^\circ \)
(two decimal places)

3. Refer to the figure. B, C and D are three points in the same horizontal plane so that \( BD = CD = d \) and \( \angle CBD = \theta \). AB is perpendicular to the plane. From C the angle of elevation of A is \( \alpha \).
(a) Express \( \hat{D} \) in terms of \( \theta \).
(b) Prove that: \( AB = 2d \cos \theta \tan \alpha \)
(c) If \( d = \sqrt{2} \) units; \( \alpha = 30^\circ \) and \( \theta = 75^\circ \) calculate AB without the use of a calculator.

4. AC represents a vertical tower which is perpendicular to the horizontal plane BCD at C. AB is 2 units. \( \angle BCD = 90^\circ - \theta \), \( \angle BDC = 2\theta \) and \( \angle BAC = \theta \).
(a) Determine BC in terms of \( \theta \).
(b) Show that BD = 1 unit.
(c) If \( AD = \sqrt{3} \) units, calculate the size of \( \angle ABD \) without using a calculator.

**REVISION EXERCISE**

1. \( 17 \sin \theta - 15 = 0 \), where \( \theta \in (90^\circ; 270^\circ) \). Determine without the use of a calculator and with the aid of a diagram the values of:
(a) \( \cos \theta \)  
(b) \( \cos 2\theta \)  
(c) \( \tan 2\theta \)
2. \[ 17 \sin 20 - 15 = 0, \text{ where } 20 \in (90° ; 270°) \]. Determine without the use of a calculator and with the aid of a diagram the values of:
   (a) \( \cos 2\theta \)  
   (b) \( \cos \theta \)
3. Consider each of the diagrams below and then determine the value of \( \tan \theta \) in each case.
   (a)  
   (b)  

4. If \( \cos 21° = k \) determine the following in terms of \( k \) without the use of a calculator:
   (a) \( \cos 201° \)  
   (b) \( \cos 42° \)  
   (c) \( \cos 51° \)  
   (d) \( \sin 21° \)
5. If \( \sin 116° = m \) determine the value of \( \tan 64° \) in terms of \( m \).
6. Simplify the following:
   (a) \( \frac{\sin(180° + 2x)}{\cos(180° + x)} \)  
   (b) \( \tan 45° - \sin^2(-x) \)  
   (c) \( \frac{\sin 4x}{1 - 2\cos^2 2x} \)  
   (d) \( \frac{\sin(-180° - \alpha) \cdot \tan(-\alpha) \cdot \cos(360° - \alpha)}{\cos^2(90° + \alpha) + \sin^2(180° + \alpha)} \)
7. Evaluate the following without the use of a calculator:
   (a) \( \cos 20° \cdot \cos 40° + \cos 250° \cdot \cos 310° \)  
   (b) \( 4 \sin 22.5° \cdot \cos 22.5° \)  
   (c) \( \frac{\cos^2 945° - 2 \sin 195° \cos 15°}{\sqrt{2} \cos 495°} \)  
   (d) \( \cos^2 15° - \sin^2 22.5° + \cos^2 22.5° + \sin^2 15° \)
8. (a) Prove that \( \frac{\sin 2x - \cos 2x - 1}{\sin x - \cos x} = 2 \cos x \)
   (b) For which values of \( x \) will the identity be invalid where \( x \in (-180° ; 90°) \)
9. Find the general solution of each of the following equations.
   (a) \( \sin 3\theta = -0.12 \)  
   (b) \( 3 \tan 3\theta = 12 \)  
   (c) \( 3 \sin \theta + 2 \cos \theta = 0 \)
10. Find the general solution of each of the following equations.
    (a) \( \sin 2\theta = \cos \theta \)  
    (b) \( \sin 2\theta = -\cos 2\theta \)  
    (c) \( \sin 2\theta = 2 \cos \theta \)  
    (d) \( \sin 2\theta = 2 \cos 2\theta \)  
    (e) \( \cos 2\theta = 4 \sin \theta + 3 \)
11. Solve for \( \theta \), if \( \tan^2 \theta = 4 \), and \( \theta \in (-180° ; 180°) \)
12. Solve for \( x \) if:
    \[ 2 \sin x \cdot \cos x + 2 \sin x + \cos^2 x + \cos x = 0 \]  
    for \( x \in [-180° ; 180°] \)
13. Given: \( \sin(A + B) - \sin(A - B) = 2 \cos A \sin B \).
   (a) Prove the above identity.
   (b) Use the identity to factorise: \( \sin 5x - \sin x \)
   (c) Hence, or otherwise, find the general solution for \( x \) if \( \sin 5x - \sin x = 0 \)
14. G, K and J are three points on the same horizontal plane. HG and IJ are the two vertical poles. Wires are strung from K to the tops of the poles. The angle of elevation of H from K is \( \alpha \) and the angle of elevation of I from K is \( \beta \). The length of wire KI is \( p \) metres.

\[ \hat{GKJ} = x. \]

(a) Show that \( KJ \cos \beta = \beta \)

(b) Show that \( GJ = \frac{p \cos \beta \sin(x + \theta)}{\sin \theta} \)

(c) Prove that \( HG = \frac{p \cos \beta \sin(x + \theta) \tan \alpha}{\sin \theta} \)

**SOME CHALLENGES**

1. If \( \sin 23^\circ = p \) determine the value of \( \cos 44^\circ \) in terms of \( p \).
2. Simplify \( P \) as far as possible if \( P = \cos(180^\circ - x) - 2\sin(210^\circ + x) \).
3. If \( \cos x = 2m \) and \( \cos 2x = 7m \), determine the value(s) of \( m \).
4. Prove that \( 4\sin^2 \theta - 11\cos \theta = 1 \) has no real solution for \( \theta \in [90^\circ ; 270^\circ] \)
5. Find the value of \( \tan \theta \) if the distance between the point \((\cos \theta ; \sin \theta)\) and \((4;5)\) is \( \sqrt{42} \).
6. The depth, \( d \) metres, of water in a harbour on a certain day is given by \( d = 5 - 3\sin(15t) \) where \( t \) is the number of hours after 12 (midnight). Calculate the times on this day when the depth of the water in the harbour was 6.5 metres.
7. Consider the expression \( 2\sin x + 2\cos x \):

   (a) Prove that \( (2\sin x + 2\cos x)^2 = 4\sin 2x + 4 \)

   (b) Hence determine the maximum value of \( 2\sin x + 2\cos x \)

   (c) What is the corresponding value(s) of \( x \) for the maximum value in 7(b) if \( 0^\circ \leq x \leq 180^\circ \) ?
8. Given that \( \sqrt{5}\tan A = -2 \) and \( \cos B = \frac{8}{17} \) in \( \triangle ABC \)

   (a) State why we may assume that \( C \) is acute.

   (b) Determine the value of \( \sin C \).
9. \( \triangle PQR \) is an equilateral triangle with sides of length \( x \) units. Its sides are extended by their own length to points D, E and F, D, E and F are joined to form \( \triangle DEF \), which is also an equilateral triangle. \( \hat{FE} = \theta \)

Show that \( \theta = 41^\circ \) (rounded off to the nearest degree)
CHAPTER 5 – ALGEBRA

A function of the form \( f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n \) \((a_0 \neq 0)\) where \( n \) is a whole number, is called a polynomial of degree \( n \). The real numbers \( a_0, a_1, a_2, \ldots, a_n \) are called the coefficients of \( x \).

Examples of polynomials include \( 2x + 3 \), \( 5x^2 - 2x + 1 \), \( x^3 - 7x - 6 \) and \( 2x^4 - 3x^3 - x \).

A linear polynomial such as \( 2x + 3 \) is a polynomial of degree 1.

A quadratic such as \( 5x^2 - 2x + 1 \) is a polynomial of degree 2.

A cubic polynomial such as \( x^3 - 7x - 6 \) is a polynomial of degree 3.

Expressions such as \( \frac{1}{x^2} + 3x \) and \( \sqrt{3}x + 5 \) are not polynomials.

FACTORIZING A CUBIC POLYNOMIAL

There are two methods of factorising cubic polynomials that you are familiar with at this stage:

(a) Sum and difference of two cubes
(b) Grouping in pairs

FACTORIZATION OF THE SUM AND DIFFERENCE OF TWO CUBES

Factorising cubic polynomials which have a term in \( x^3 \) and a constant term can be factorised as follows:

\[
x^3 + a^3 = (x + a)(x^2 - ax + a^2) \quad \text{and} \quad x^3 - a^3 = (x - a)(x^2 + ax + a^2)
\]

The above can also be summarized as follows:

\[
(first)^3 \pm (last)^3 = (first \pm last)((first)^2 \mp (first)\times(last) + (last)^2)
\]

EXAMPLE 1

Factorise the following:

(a) \( 8x^3 - 27 \)  
(b) \( x^3 + 64 \)

Solutions

(a) \( \sqrt[3]{8x^3} = 2x \) and \( \sqrt[3]{27} = 3 \)

\[
\therefore 8x^3 - 27 = (2x - 3)(4x^2 + 6x + 9)
\]

(b) \( \sqrt[3]{x^3} = x \) and \( \sqrt[3]{64} = 4 \)

\[
\therefore x^3 + 64 = (x + 4)(x^2 - 4x + 16)
\]

FACTORISATION BY GROUPING IN PAIRS

Grouping is a factorising method that can be used when an expression has four or more terms. A cubic polynomial has at most four terms and the terms will be grouped in pairs.

EXAMPLE 2

Factorise the following:

(a) \( 2x^3 + 4x^2 + 3x + 6 \)  
(b) \( 3x^3 + x^2 - 12x - 4 \)
Solutions

(a) Group the first two terms together and the last two terms together. Factorise each part separately and then take out a common bracket.

\[2x^3 + 4x^2 + 3x + 6\]

\[= 2x^2(x + 2) + 3(x + 2)\]

\[= (x + 2)(2x^2 + 3)\]

(b) \[3x^3 + x^2 - 12x - 4\]

\[= x^2(3x + 1) - 4(3x + 1)\]

\[= (3x + 1)(x^2 - 4)\]

\[= (3x + 1)(x - 2)(x + 2)\]

REVISION EXERCISE

Factorise the following completely:

(a) \[x^3 - 8\]  
(b) \[8x^3 + 1\]  
(c) \[x^3 + x^2 - 2x - 2\]  
(d) \[27x^3 + 27\]  
(e) \[x^3 - 3x^2 - 3x + 9\]  
(f) \[x^3 + 3x^2 + 2x + 6\]  
(g) \[-x^3 + x^2 + 9x - 9\]  
(h) \[4x^3 - x^2 - 16x + 4\]  
(i) \[27x^3 - 8\]

DIVIDING POLYNOMIALS USING LONG DIVISION

The trinomial \(3x^2 + 7x - 6\) can be factorised as follows:

\[3x^2 + 7x - 6 = (x + 3)(3x - 2)\]

If \(3x^2 - 7x - 6\) is divided by \(x + 3\) the answer (quotient) is equal to \(3x - 2\):

\[
\frac{3x^2 + 7x - 6}{x + 3} = \frac{(x + 3)(3x - 2)}{(x + 3)} = 3x - 2
\]

An alternative method of dividing polynomials is long division. This method is highly useful for factorising cubic polynomials.

Consider the following numerical example which illustrates the process of long division. Suppose that 1043 divided by 6.

\[
\begin{array}{c|cccc}
\text{Divisor} & 1 & 7 & 3 \\
\hline
\text{Dividend} & 1 & 0 & 4 & 3 \\
6 & 6 & 4 & 4 & 42 \\
\hline
\text{Quotient} & 2 & 3 & 1 & 8 \\
\text{Remainder} & 5 \\
\end{array}
\]

Note:

In this long division, an interesting property referred to as the Euclidean property emerges:

Dividend = divisor \(\times\) quotient + remainder

\[1043 = 6 \times 173 + 5\]
This process of long division can be extended to polynomial division.

**EXAMPLE 3**

(a) Divide \(3x^2 + 7x - 6\) by \(x + 3\) using long division.

Step 1: Divide the first terms of both expressions: \(\frac{3x^2}{x} = 3x\) and write it on top.

\[
x + 3 \longdiv{3x^2 + 7x - 6}
\]

3

\[x + 3 \longdiv{3x^2 + 7x - 6}
\]

Step 2: Multiply the \(3x\) with \(x + 3\): \(3x(x + 3) = 3x^2 + 9x\) and write it below.

\[
x + 3 \longdiv{3x^2 + 7x - 6}
\]

\[
\frac{3x}{x} \longdiv{3x^2 + 9x}
\]

Step 3: Subtract \(3x^2 + 9x\) from \(3x^2 + 7x\):

\[
x + 3 \longdiv{3x^2 + 7x - 6}
\]

\[
\frac{3x}{x} \longdiv{3x^2 + 9x - 2x - 6}
\]

Step 4: Bring down the \(-6\).

Now this cycle of divide, multiply, subtract and bring down will start again.

\[
x + 3 \longdiv{3x^2 + 7x - 6}
\]

\[
\frac{3x}{x} \longdiv{3x^2 + 9x - 2x - 6}
\]

Divide: \(\frac{-2x}{x} = -2\) (Write on top)

Multiply: \(-2(x + 3) = -2x - 6\) (Write below)

Subtract: \(-2x - 6 - (-2x - 6) = -2x - 6 + 2x + 6 = 0\)

\[
\frac{3x^2 + 7x - 6}{x + 3} = (x + 3)(3x - 2) + 0
\]

\[
\therefore 3x^2 + 7x - 6 = (x + 3)(3x - 2)
\]

(b) Divide \(x^3 - 16x + 4\) by \(x - 2\) using long division.

Before continuing, take careful note that the term \(0x^2\) has been inserted. This makes the subtraction part easier.

\[
x - 2 \longdiv{x^3 + 0x^2 - 16x + 4}
\]

Divide: \(\frac{x^3}{x} = x^2\) (Write on top)

Multiply: \(x^2(x - 2) = x^3 - 2x^2\) (Write below)

Subtract: \(x^3 + 0x^2 - (x^3 - 2x^2)\)

\[
\begin{align*}
2x^2 - 16x
\end{align*}
\]

Bring down \(-16x\)
Repeat the above process twice.

\[
\begin{array}{r}
\frac{x^2 + 2x - 12}{x^3 - 2x^2} \\
\downarrow \\
\frac{x^3 - 2x^2}{2x^2 - 16x} \\
\downarrow \\
\frac{2x^2 - 16x}{2x^2 - 4x} \\
\downarrow \\
\frac{2x^2 - 4x}{-12x + 4} \\
\downarrow \\
\frac{-12x + 4}{-12x + 24} \\
\downarrow \\
\frac{-12x + 24}{-20}
\end{array}
\]

Dividend = divisor \times quotient + remainder

\[
\therefore x^3 - 16x + 4 = (x - 2)(x^2 + 2x - 12) - 20
\]

More generally, we can consider the division of a polynomial \( f(x) \) by \( ax - b \) as follows:
Let Q(x) be the quotient and R the remainder, then: \( f(x) = (ax - b)Q(x) + R \)

**EXERCISE 1**

1. In the following, \( f(x) \) is divided by \( g(x) \). Find the remainder by using the process of long division:
   (a) \( f(x) = x^3 + 2x^2 + 1 \) and \( g(x) = x - 2 \)
   (b) \( f(x) = x^3 - 2x - 2 \) and \( g(x) = x + 1 \)
   (c) \( f(x) = x^3 - 7x - 6 \) and \( g(x) = x + 1 \)
   (d) \( f(x) = 6x^3 - 4x^2 + x - 3 \) and \( g(x) = 3x + 1 \)

2. Divide \( 4x^2 - x + 5 \) by \( x + 2 \) using long division.
3. Divide \( 2x^2 + 7x - 5 \) by \( 2x - 1 \) using long division.
4. Divide \( x^3 - x^2 + 4x - 3 \) by \( x - 3 \) using long division.
5. Divide \( x^3 - 12x + 16 \) by \( x - 2 \) using long division.
6. Determine the remainder of \( (4x^3 - 6x^2 + 11) + (2x - 1) \)
7. Show that \( x + 4 \) is a factor of \( x^3 + 3x^2 + 16 \) (Hint: show that the remainder is zero)

**THE REMAINDER AND FACTOR THEOREM**

**The Remainder Theorem**

If a polynomial \( f(x) \) is divided by a linear polynomial \( ax - b \), then the remainder is \( f\left(\frac{b}{a}\right) \)

**Proof** (Not for exam purposes)

\[
f(x) = (ax - b)Q(x) + R \text{ holds for all values of } x \text{ and therefore also for } x = \frac{b}{a}.
\]

\[
\therefore f\left(\frac{b}{a}\right) = \left(a\left(\frac{b}{a}\right) - b\right)Q\left(\frac{b}{a}\right) + R
\]

\[
\therefore f\left(\frac{b}{a}\right) = (b-b)Q\left(\frac{b}{a}\right) + R
\]

\[
\therefore f\left(\frac{b}{a}\right) = 0.Q\left(\frac{b}{a}\right) + R = R
\]
From the remainder theorem we know that if:

(a) \( f(x) \) is divided by \( x + 1 \) then the remainder is \( f(-1) \)
(b) \( f(x) \) is divided by \( x - 3 \) then the remainder is \( f(3) \)
(c) \( f(x) \) is divided by \( 3x - 2 \) then the remainder is \( f\left(\frac{2}{3}\right) \)

EXAMPLE 4

Find the remainder when \( f(x) = x^3 - 16x + 4 \) is divided by \( x - 2 \).

Solution

From the previous example we showed that the remainder is \(-20\) using long division. The remainder theorem can also be used to determine the remainder.

Let \( x - 2 = 0 \)

\( \therefore x = 2 \)

\( \therefore \text{Rem} = f(2) = (2)^3 - 16(2) + 4 = 8 - 32 + 4 = -20 \)

The remainder is \(-20\) when \( f(x) \) is divided by \( x - 2 \).

EXAMPLE 5

(a) The remainder when \( f(x) = -2x^3 + ax^2 - 4x + 3 \) is divided by \( x + 3 \) is 15. Determine the value of \( a \).

Solution

Let \( x + 3 = 0 \)

\( \therefore x = -3 \)

Rem = \( f(-3) = -2(-3)^3 + a(-3)^2 - 4(-3) + 3 \) and Rem = 15

\( \therefore -54 + 9a + 12 + 3 = 15 \)

\( \therefore 9a = 54 \)

\( \therefore a = 6 \)

(b) The remainder when \( g(x) = 16ax^3 - 2bx + 5 \) is divided by \( x + 1 \) is \(-9\) while the remainder is 6 when divided by \( 2x - 1 \). Determine the values of \( a \) and \( b \).

Solution

Rem = \( g(-1) = 16a(-1)^3 - 2b(-1) + 5 \) and Rem = \(-9\)

\( \therefore -16a + 2b + 5 = -9 \)

\( \therefore -16a + 2b = -14 \)

\( \therefore 8a - b = 7 \)

and Rem = \( g\left(\frac{1}{2}\right) = 16a\left(\frac{1}{2}\right)^3 - 2b\left(\frac{1}{2}\right) + 5 \) and Rem = 6

\( \therefore 2a - b + 5 = 6 \)

\( \therefore 2a - b = 1 \)
Solve the two equations simultaneously:

\[ 8a - b = 7 \quad \text{A} \quad 2a - b = 1 \quad \text{B} \]

\[ 8a - b = 7 \quad \text{A} \]
\[ -2a + b = -1 \quad \text{B} \times -1 \]
\[ \therefore 6a = 6 \]
\[ \therefore a = 1 \]
\[ \therefore 2(1) - b = 1 \]
\[ \therefore -b = -1 \]
\[ \therefore b = 1 \]

**EXERCISE 2**

1. In the following, \( f(x) \) is divided by \( g(x) \). Find the remainder by using the remainder theorem:

(a) \( f(x) = x^3 + 2x^2 + 1 \) \( g(x) = x - 2 \)

(b) \( f(x) = x^3 - 2x - 2 \) \( g(x) = x + 1 \)

(c) \( f(x) = x^3 - 7x - 6 \) \( g(x) = x + 1 \)

(d) \( f(x) = 4x^2 - x + 5 \) \( g(x) = x + 2 \)

(e) \( f(x) = 6x^3 - 4x^2 + x - 3 \) \( g(x) = 3x + 1 \)

2. Find the remainder when \( f(x) = \frac{1}{2}x^3 - 3x^2 - 4x \) is divided by \( x - 4 \).

3. The remainder when \( f(x) = x^3 - ax^2 - x + 2a \) is divided by \( x + 1 \) is \(-4\). Determine the value of \( a \).

4. Find the value of \( a \) if \( f(x) = x^3 + ax^2 - x + 5 \) is divided by \( x - 2 \) to give a remainder of 23.

5. If \( f(x) = x^5 - 2x^4 - x^3 + 3x^2 + px + 6 \) is divided by \( x + 1 \), the remainder is 2. Find the value of \( p \).

6. When the function \( x^2 + qx + r \) is divided by \( x - 1 \) and \( x + 2 \) the remainders are 8 and 5 respectively. Determine the values of \( q \) and \( r \).

7. When \( x^3 + mx^2 + nx + 1 \) is divided by \( x - 2 \) the remainder is 9; when divided by \( x + 3 \) the remainder is 19. Find the values of \( m \) and \( n \).

8. If \( f(x) = ax^3 + bx^2 - 3x + 6 \) is divided by \( x + 1 \), the remainder is 12. If \( f(x) \) is divided by \( 2x - 1 \), the remainder is \( 4 \frac{1}{2} \). Calculate the value of \( a \) and \( b \).

9. Show that \( x + 4 \) is a factor of \( x^3 + 3x^2 + 16 \) without using long division.

**THE FACTOR THEOREM**

When two polynomials are divided and the remainder is zero, the divisor is called a factor of the dividend. The factor theorem states this as follows:

If \( f(x) \) is a polynomial such that \( f\left(\frac{b}{a}\right) = 0 \), then \( ax - b \) is a factor of \( f(x) \)

Conversely: If \( ax - b \) is a factor of \( f(x) \) then \( f\left(\frac{b}{a}\right) = 0 \)
The factor theorem can be used to finding linear factors of cubic polynomials. There are two additional methods available for factorising cubic polynomials. These methods will now be discussed in detail. The methods are:
(a) Long division  
(b) Inspection

EXAMPLE 6

(a) Show that $x + 1$ is a factor of $f(x) = 2x^3 - 2x^2 - 10x - 6$
(b) Factorise $f(x)$ completely using long division.
(c) Factorise $f(x)$ completely using inspection.

Solutions

(a) You have to show that the remainder equals zero.
   Let $x + 1 = 0$
   $\therefore x = -1$
   $\therefore$ Rem $= f(-1) = 2(-1)^3 - 2(-1)^2 - 10(-1) - 6$
   $= -2 - 2 + 10 - 6 = 0$
   $\therefore x + 1$ is a factor of $f(x)$

(b) $f(x) = 2x^3 - 2x^2 - 10x - 6$
   $\therefore x + 1$ $\frac{2x^3 - 4x - 6}{2x^3 - 2x^2 - 10x + 6}$
   \[
   \begin{array}{c|cccc}
   & 2x^2 & -4x & -6 \\
   \hline
   x+1 & 2x^3 & -2x^2 & -10x & +6 \\
   & 2x^3 & +2x^2 & & \\
   \hline
   & & -4x^2 & -10x & \\
   & & -4x^2 & -4x & \\
   \hline
   & & & -6x & +6 \\
   & & & -6x & +6 \\
   \hline
   & & & & 0
   \end{array}
   \]
   $\therefore f(x) = 2x^3 - 2x^2 - 10x - 6 = (x + 1)(2x^2 - 4x - 6) + 0$
   $= (x + 1)2(x^2 - 2x - 3)$
   $= 2(x + 1)(x + 1)(x - 3)$
   $= 2(x + 1)^2(x - 3)$

(c) Only one of the factors is known to us. The original function is a cubic polynomial and therefore the other factor will be a quadratic polynomial.
   $\therefore f(x) = (x + 1)(ax^2 + bx + c)$
   Multiply the **first terms** of each bracket: $x \times ax^2 = ax^3$.
   Consider the original function and notice that $ax^3$ has to equal $2x^3$
   By comparing the coefficients of the two it is easy to deduce that $a = 2$
   $ax^3 = 2x^3$
   $\therefore a = 2$
   Multiply the **last terms** of each bracket: $1 \times c = 1c$.
   Consider the original function and notice that $1c$ has to equal the constant term $-6$.
   $\therefore 1c = -6$
   $\therefore c = -6$
Fill in what you have: \((x + 1)(2x^2 + bx - 6)\)

Consider the products that will produce the terms in \(x^3\) and compare them to the original term in \(x^2\) which is \(-2x^2\).

\[f(x) = 2x^3 - 2x^2 - 10x - 6\]

The coefficients must be equal. Multiply as shown below.

\[
\begin{align*}
(x + 1)(2x^2 + bx - 6) & \\
\therefore 2x^2 + bx^2 & = -2x^2 \\
\therefore x^2(2 + b) & = -2x^2 \\
\therefore 2 + b & = -2 \quad \text{equate the coefficients} \\
\therefore b & = -4 \\
\therefore f(x) & = (x + 1)(2x^2 - 4x - 6) \\
\therefore f(x) & = (x + 1)2(x^2 - 2x - 3) \\
\therefore f(x) & = 2(x + 1)(x + 1)(x - 3) \\
\therefore f(x) & = 2(x + 1)^2(x - 3)
\end{align*}
\]

**EXAMPLE 7**

Factorise \(x^3 - 12x - 16\) completely.

**Solution**

Use the **factor theorem** to determine a factor of the cubic polynomial. Always consider factors of the last term. Hence consider the factors of \(-16\).

\[
\begin{align*}
\therefore x & = \pm 1 \text{ or } x = \pm 2 \text{ or } x = \pm 4 \text{ or } x = \pm 8 \text{ or } x = \pm 16 \\
\text{If } x & = 1 \text{ then Rem } = (1)^3 - 12(1) - 16 = -27 \quad \therefore x - 1 \text{ is not a factor} \\
\text{If } x & = -1 \text{ then Rem } = (-1)^3 - 12(-1) - 16 = 5 \quad \therefore x + 1 \text{ is not a factor} \\
\text{If } x & = 2 \text{ then Rem } = (2)^3 - 12(2) - 16 = -32 \quad \therefore x - 2 \text{ is not a factor} \\
\text{If } x & = -2 \text{ then Rem } = (-2)^3 - 12(-2) - 16 = 0 \quad \therefore x + 2 \text{ is a factor} \\
\therefore x^3 - 12x - 16 & = (x + 2)(ax^2 + bx + c)
\end{align*}
\]

Multiply the first terms of each bracket: \(\therefore ax^3 = x^3 \quad \therefore a = 1\)

Multiply the last terms of each bracket: \(\therefore 2c = -16 \quad \therefore c = -8\)

Fill in what you’ve determined: \((x + 2)(x^2 + bx - 8)\)

Consider the products that will produce the terms in \(x^2\):

\[
\begin{align*}
(x + 2)(x^2 + bx - 8) & \\
\therefore 2x^2 + bx^2 & = 0x^2 \\
\therefore 2 + b & = 0 \quad \text{equate the coefficients} \\
\therefore b & = -2
\end{align*}
\]

Write the expression as \(x^3 + 0x^2 - 12x - 16\).
\[ x^3 - 12x - 16 = (x + 2)(x^2 - 2x - 8) \]
\[ x^3 - 12x - 16 = (x + 2)(x + 2)(x - 4) \]
\[ x^3 - 12x - 16 = (x + 2)^2(x - 4) \]

**EXAMPLE 8**

Determine the value of \( p \) if \( x + 2 \) is a factor of \( x^3 + px + (3 - p) \).

**Solution**

Let \( x + 2 = 0 \)
\[ \therefore x = -2 \]
\[ \therefore \text{Rem} = (-2)^3 + p(-2) + (3 - p) \]
\[ = -8 - 2p + 3 - p \]
\[ = -3p - 5 \]

From the remainder theorem, it is clear that the remainder is \(-3p - 5\). However, the remainder is 0 since the divisor is a factor.

\[ \therefore -3p - 5 = 0 \]
\[ \therefore -3p = 5 \]
\[ \therefore p = -\frac{5}{3} \]

**EXERCISE 3**

1. Determine the values of \( a \), \( b \) and \( c \) in each case
   (a) \( x^3 - 2x^2 + 4x - 8 = (x + 2)(ax^2 + bx + c) \)
   (b) \( 2x^3 + x^2 - 2x - 1 = (2x + 1)(ax^2 + bx + c) \)
   (c) \( 2x^3 + 5x^2 - x - 6 = (x + 2)(ax^2 + bx + c) \)
   (d) \( 2x^3 - x^2 - 8x + 4 = (2x - 1)(ax^2 + bx + c) \)

2. Show that \( x - 1 \) is a factor of \( f(x) = x^3 - 7x + 6 \) and hence factorise \( f(x) \) completely.
3. Show that \( 2x + 1 \) is a factor of \( f(x) = 2x^3 + x^2 - 8x - 4 \) and hence factorise \( f(x) \) completely.
4. Factorise the following polynomials completely:
   (a) \( x^3 - x^2 - 22x + 40 \)
   (b) \( x^3 + 2x^2 - 5x - 6 \)
   (c) \( 3x^3 - 7x^2 + 4 \)
   (d) \( x^3 - 19x + 30 \)
   (e) \( x^3 - x^2 - x - 2 \)

5. If \( 2x + 1 \) is a factor of the expression \( x^3 - 2ax^2 + 4x - 1 \) find the value of \( a \).
6. If \( x - 2 \) is a factor of \( x^3 + mx + 6 \), determine the value of \( m \).
7. If \( x - 1 \) is a factor of \( x^4 - 15x^2 + bx + 24 \), determine the value of \( b \).
8. If \( x + 2 \) is a factor of \( x^5 + x^4 + ax^3 - 2x^2 + mx + 14 \), determine \( a \) in terms of \( m \).
9. If \( x - p \) is a factor of the expression \( x^3 - 3x^2 - 5px - 9x \) find the possible values of \( p \).
10. Find \( m \) if \( x - 3 \) and \( x + 2 \) are factors of \( x^3 + m^2x^2 - 11x - 15m \).
11. Given that the expression \( 2x^3 + ax^2 + b \) is divisible by \( x + 1 \) and that there is a remainder of 16 when it is divided by \( x - 3 \), find the values of \( a \) and \( b \).
SOLVING CUBIC EQUATIONS

The standard form of a cubic equation is \( ax^3 + bx^2 + cx + d = 0 \). Cubic equations have at most three real solutions (roots).

**EXAMPLE 9** (Equations of the form \( ax^3 + bx^2 + cx = 0 \))

Solve the following equation: \( 2x^3 + 5x^2 = 3x \)

**Solution**

\[
2x^3 + 5x^2 - 3x = 0 \quad \text{(write the cubic equation in standard form)}
\]

\[
x(2x^2 + 5x - 3) = 0
\]

\[
x = 0 \quad \text{or} \quad 2x^2 + 5x - 3 = 0
\]

\[
2x^2 + 5x - 3 = 0 \text{ is quadratic equation and can be solved further by either factorising or } \text{by using the quadratic formula.}
\]

\[
x = 0 \quad \text{or} \quad 2x^2 + 5x - 3 = 0
\]

\[
x = 0 \quad \text{or} \quad (2x - 1)(x + 3) = 0
\]

\[
x = 0 \quad \text{or} \quad 2x = 1 \text{ or } x = -3
\]

\[
x = 0 \quad \text{or} \quad x = \frac{1}{2} \text{ or } x = -3
\]

\[
x = 0 \quad \text{or} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
x = 0 \quad \text{or} \quad x = \frac{-5 \pm \sqrt{49}}{4}
\]

\[
x = 0 \quad \text{or} \quad x = \frac{1}{2} \text{ or } x = -3
\]

**EXAMPLE 10** (Equations of the form \( ax^3 + d = 0 \))

Solve the following equations:

(a) \( x^3 - 8 = 0 \) \quad (b) \( 8x^3 + 27 = 0 \) \quad (c) \( (x - 2)^3 + 125 = 0 \)

**Solutions**

All of the above examples are cubic expressions with a term in \( x^3 \) and a constant term. These equations may be solved as follows:

(a) \( x^3 - 8 = 0 \) \quad (b) \( 8x^3 + 27 = 0 \) \quad (c) \( (x - 2)^3 + 125 = 0 \)

\[
x^3 = 8 \quad \text{or} \quad 8x^3 = -27 \quad \text{or} \quad (x - 2)^3 = -125
\]

\[
x = \sqrt[3]{8} = 2 \quad \text{or} \quad x = \frac{-27}{8} \quad \text{or} \quad x - 2 = -\sqrt[3]{125}
\]

\[
x = \frac{-27}{8} \quad \text{or} \quad x = \frac{-5 + 2}{3} = \frac{3}{2}
\]
EXAMPLE 11  (Equations of the form \( ax^3 + bx^2 + cx + d = 0 \))

Solve for \( x \): \( x^3 - 12x + 16 = 0 \)

Solution

Method 1  (Long division)

Consider the factors: \( x = \pm 1 \) or \( x = \pm 2 \) or \( x = \pm 4 \) or \( x = \pm 8 \) or \( x = \pm 16 \)

If \( x = 1 \) then the rem = \((1)^3 - 12(1) + 16 = 5 \) \( \therefore x - 1 \) is not a factor

If \( x = -1 \) then the rem = \((-1)^3 - 12(-1) + 16 = 27 \) \( \therefore x + 1 \) is not a factor

If \( x = 2 \) then the rem = \((2)^3 - 12(2) + 16 = 0 \) \( \therefore x - 2 \) is a factor

\[
\begin{align*}
x^3 + 0x^2 - 12x + 16 &= 0 \\
\therefore (x - 2)(x^2 + 2x - 8) &= 0 \\
\therefore (x - 2)(x + 4)(x - 2) &= 0 \\
\therefore x = 2 \text{ or } x = -4
\end{align*}
\]

Method 2  (Inspection)

\( x^3 - 12x + 16 = 0 \)  (write the cubic equation in standard form)

Factorise the left side by first finding a factor using the factor theorem and then factorise further using inspection.

Consider the factors: \( x = \pm 1 \) or \( x = \pm 2 \) or \( x = \pm 4 \) or \( x = \pm 8 \) or \( x = \pm 16 \)

If \( x = 1 \) then the rem = \((1)^3 - 12(1) + 16 = 5 \) \( \therefore x - 1 \) is not a factor

If \( x = -1 \) then the rem = \((-1)^3 - 12(-1) + 16 = 27 \) \( \therefore x + 1 \) is not a factor

If \( x = 2 \) then the rem = \((2)^3 - 12(2) + 16 = 0 \) \( \therefore x - 2 \) is a factor

\[
\begin{align*}
x^3 - 12x + 16 &= 0 \\
\therefore x^3 - 12x + 16 &= (x - 2)(ax^2 + bx + c) \\
\therefore ax^3 &= x^3 \quad \therefore a = 1 \\
\therefore -2c &= 16 \quad \therefore c = -8 \\
\text{Fill in what you have: } (x - 2)(1x^2 + bx - 8) \\
\text{Consider the products that will produce the terms with a } x^2: \\
(x - 2)(x^2 + bx - 8) \quad \text{The expression } x^3 - 12x + 16 \text{ does not have } x^2 \text{ term.} \\
\therefore x^3 + 0x^2 - 12x + 16 \\
\therefore -2x^2 + bx^2 = 0x^2 \\
\therefore -2 + b = 0 \\
\therefore b = 2 \\
\therefore x^3 - 12x + 16 = (x - 2)(x^2 + 2x - 8) \\
\therefore (x - 2)(x - 2)(x + 4) = 0 \\
\therefore (x - 2)(x - 2)(x + 4) = 0 \\
\therefore x = 2 \text{ or } x = -4
\end{align*}
\]
EXERCISE 4

1. Solve for \( x \):
   (a) \((x - 3)(x + 2)(2x + 1) = 0\) \( \quad \) (b) \(x(3x + 2)(x - 2) = 0\)
   (c) \((x - 3)^2(3x + 1) = 0\) \( \quad \) (d) \(x^3 + 1 = 0\)
   (e) \(3x^3 - 81 = 0\) \( \quad \) (f) \((x - 3)^3 - 64 = 0\)
   (g) \(\frac{1}{3}x^3 + 9 = 0\) \( \quad \) (h) \(x^3 - x^2 - 12x = 0\)
   (i) \(x^3 + 2x^2 - 4x = 0\) \( \quad \) (j) \(x^3 - 3x^2 - x + 6 = 0\)
   (k) \(2x^3 - 12x^2 - x + 6 = 0\) \( \quad \) (l) \(2x^3 - x^2 - 8x + 4 = 0\)
   (m) \(x^3 + x^2 - 2 = 0\) \( \quad \) (n) \(x^3 = 16 + 12x\)
   (o) \(x^3 = 27x\) \( \quad \) (p) \(x^3 + 3x^2 = 20x + 60\)

2. Show that \(x - 3\) is a factor of \(f(x) = x^3 - x^2 - 5x + 3\) and hence solve \(f(x) = 0\)
3. Show that \(2x - 1\) is a factor of \(g(x) = 4x^3 - 8x^2 - 2 + x\) and hence solve \(g(x) = 0\)
4. Determine the coordinates of the \(x\)-intercepts of the function \(f(x) = x^3 - 4x^2 - 3x + 18\)

REVISION EXERCISE

1. Find the remainder when \(f(x) = -9x^2 - 3x + 1\) is divided by \(3x + 2\).
2. Calculate \(R\) if \(x^3 - 3x^2 + 7 = (x - 2)Q(x) + R\)
3. Show that \(x - 3\) is a factor of \(g(x) = x^3 - 4x^2 + 4x - 3\) and hence solve \(g(x) = 0\)
4. Show that \(x + 2\) is a factor of \(g(x) = -x^3 + 5x + 2\) and hence factorise \(g(x)\) completely.
5. \(x - m\) is a factor of \(f(x) = mx^2 - 12x + 16\). Determine the value(s) of \(m\).
6. When the function \(4x^2 + 2qx + r + q\) is divided by \(2x - 1\) and \(x + 2\) the remainders are \(-1\) and \(9\) respectively. Determine the values of \(q\) and \(r\).
7. Factorise the following cubic polynomials completely:
   (a) \(x - 9x^3\) \( \quad \) (b) \(x^3 - 5x^2 + 6x\) \( \quad \) (c) \(2x^3 - 3x^2 - 4x + 6\)
   (d) \(-x^3 - 4x^2 + 3\) \( \quad \) (e) \(x^3 - 4x^2 - 9x + 36\) \( \quad \) (f) \(-2x^3 - 54\)
8. Solve the following equations:
   (a) \(x^3 + 3 = 2x^2\) \( \quad \) (b) \(x^3 - 12x^2 + 36x = 0\) \( \quad \) (c) \(-2x^3 + x^2 - 8x + 4 = 0\)
9. Consider the two given functions: \(f(x) = 2x^3 - x^2 - 4x - 7\) and \(g(x) = x^3 - 3x - 8\) Determine the values of \(x\) for which these two graphs will intersect.
10. Determine the \(x\)-intercepts of the following functions:
    (a) \(g(x) = x^3 - 3x^2 + 3x - 1\) \( \quad \) (b) \(h(x) = 8 - x^3\)
SOME CHALLENGES

1. Given that the expressions $x^3 - 4x^2 + x + 6$ and $x^3 - 3x^2 + 2x + k$ have a common factor, find the possible values of $k$.

2. The expression $px^3 - 8x^2 + qx + 6$ is exactly divisible by $x^2 - 2x - 3$. Find the values of $p$ and $q$.

3. The polynomials $x^3 + 4x^2 - 2x + 1$ and $x^3 + 3x^2 - x + 7$ leave the same remainder when divided by $x - a$. Determine the possible values of $a$.

4. Find the remainder when $f(x) = x^3 - 3x^2 + 5x - 4$ is divided by $x^2 - x - 2$.

5. $f(x)$ is divisible by $x^2 - 5x - 6$ and $g(x)$ is divisible by $x^2 - 2x - 3$.
   (a) Determine the values of $f(6)$ and $g(6)$.
   (b) State the common factor for the two functions.

6. (a) Factorise $x^3 - 2x^2 - x + 2$
   (b) Hence solve the inequality $x^2 - 2x^2 - x + 2 > 0$ using the number line method.

7. Solve the following equation: $x^3 + 2x - \frac{36}{x^2 + 2x} = 9$

INVESTIGATION

The aim of this investigation is to discover that the second factor (trinomial) of any sum or difference of two cubes of the form $(ax)^3 \pm b^3$ can never equal zero for all non-zero real numbers $a$ and $b$.

1. Given: $f(x) = x^3 - 1$
   (a) Factorise $f(x)$
   (b) Hence, show that $f(x) = 0$ has only one root (solution).

2. Given: $g(x) = x^3 - 8$, $h(x) = x^3 - 27$ and $k(x) = x^3 - 64$:
   (a) Factorise $g(x)$, $h(x)$ and $k(x)$.
   (b) Hence, show that $g(x) = 0$, $h(x) = 0$ and $k(x) = 0$ have only one root (solution).

3. Show that for any non-zero real number $b$, $x^3 - b^3 = (x-b)(x^2 + bx + b^2) = 0$ has only one real solution, by showing that $x^2 + bx + b^2 = 0$ has no solution.

4. Hence or otherwise prove that $x^3 + b^3 = (x + b)(x^2 - bx + b^2) = 0$ also has only one real solution for any real number $b$.

5. (a) Factorise $(ax)^3 \pm b^3$
   (b) Hence or otherwise prove that for all real numbers $a$ and $b$, $(ax)^3 \pm b^3 = 0$ has only one real solution.
CHAPTER 6 - DIFFERENTIAL CALCULUS

Calculus is probably one of the greatest creations of human thought. It has become a powerful tool for solving important problems in the world of mathematics and science. It was during the seventeenth century that calculus was invented as a means of investigating problems that involve motion. The pioneers of calculus were the great mathematicians, Newton, Leibniz and Cauchy.

Modern day applications of calculus include investigating the rate of growth of populations, predicting the outcome of chemical reactions, measuring instantaneous changes in electrical current, describing the behaviour of atomic particles, estimating tumour shrinkage in radiation therapy, forecasting economic profits and losses, determining the spread of epidemics, examining the impact of motor vehicle emissions on ozone depletion and analyzing vibrations in mechanical systems.

When Mark Shuttleworth’s rocket was launched into space, many changes occurred rapidly. The rocket gained altitude as it accelerated to higher speeds. Its mass decreased as the fuel was burned up. Mark felt an increasing pressure due to the acceleration of the rocket. As the distance from the earth got larger, so did his weight decrease. The values of many variables changed dramatically during this time period.

Calculus helps us to make sense of all of these changes. It is essentially the mathematics of change. Wherever there is motion or growth or where variations in one quantity produce changes in another, calculus helps us to understand the changes that occur.

**THE CONCEPT OF A LIMIT**

The starting point for understanding calculus begins with the concept of a limit. From the idea of a limit, the whole of calculus develops into a masterpiece of mathematical precision. Let’s now focus on this fascinating concept of a limit and see where this all leads to!

Consider the graph of the function \( f(x) = x^2 + 1 \)

Clearly, if we select any value for \( x \), say, \( x = 2 \), then the corresponding \( y \)-value can be calculated as follows:

\[
 f(2) = (2)^2 + 1 = 5
\]

We now consider values of \( x \) to the left of 2 and to the right of 2. If these values of \( x \) are substituted into the equation of the quadratic function, it will be possible to determine to which \( y \)-value the graph approaches.
\[ f(x) = 2x \]

The limit of the function is therefore the value of \( y \) to which the graph approaches as the values of \( x \) approach a certain value from both the left and right. In advanced courses in calculus, the limit concept will be explored in greater detail. It is sufficient for our purposes to understand just the basics of the limit concept. We will now discuss some further examples involving limits.
EXAMPLE 1

Determine the following limits:

(a) \[ \lim_{x \to 1} (2x^2 + 4) = (2(1)^2 + 4) = 6 \]

(b) \[ \lim_{x \to 2} \frac{x^2 - 4}{x + 1} = \frac{(2)^2 - 4}{2 + 1} = 0 \]

(c) \[ \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \]

In this example, it is first necessary to simplify the expression before determining the limit, as the denominator will be zero if \( x = 1 \) and division by zero is undefined.

\[ \therefore \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x + 1)(x - 1)}{(x - 1)} = \lim_{x \to 1} (x + 1) = 2 \]

Notice:

The graph of \( y = \frac{x^2 - 1}{x - 1} \) represents a linear function where \( x = 1 \) is excluded from the domain.

\[ y = \frac{x^2 - 1}{x - 1} = \frac{(x + 1)(x - 1)}{(x - 1)} = x + 1 \quad \text{where} \quad x \neq 1 \]

As the values of \( x \) approach 1 from the left and right, the \( y \)-values approach 2.

It is important to note that even though \( f(1) \) is undefined it is clear that \( \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2 \).

The \( y \)-value approached is the limit even though the function value is undefined at that particular value of \( x \).

Remember the following factorisation principles:

\[
\begin{align*}
a^2 - b^2 &= (a + b)(a - b) & (b - a) &= -(a - b) \\
a^2 - 2ab + b^2 &= (a - b)^2 & a^2 + 2ab + b^2 &= (a + b)^2 \\
a^3 - b^3 &= (a - b)(a^2 + ab + b^2) & a^3 + b^3 &= (a + b)(a^2 - ab + b^2)
\end{align*}
\]
(d) \[ \lim_{h \to 0} \frac{(2h + 3)^2 - 9}{h} \]
\[= \lim_{h \to 0} \frac{4h^2 + 12h + 9 - 9}{h} \]
\[= \lim_{h \to 0} \frac{4h^2 + 12h}{h} \]
\[= \lim_{h \to 0} \frac{4h(h + 3)}{h} \]
\[= \lim_{h \to 0} 4(h + 3) \]
\[= 4(0 + 3) \]
\[= 12 \]

(e) \[ \lim_{x \to 3} \frac{x^3 - 27}{3 - x} \]
\[= \lim_{x \to 3} \frac{(x - 3)(x^2 + 3x + 9)}{-(x - 3)} \]
\[= \lim_{x \to 3} -(x^2 + 3x + 9) \]
\[= -((3)^2 + 3(3) + 9) \]
\[= -27 \]

(f) \[ \lim_{x \to 2} 7 \]
In this example, the graph of \( f(x) = 7 \) is a horizontal line. This means that for any point on this line, the y-values will always be 7. This means that if the values of x approach 2 from the left or right, the graph will still approach a y-value of 7.
\[\therefore \lim_{x \to 2} 7 = 7 \]

(g) \[ \lim_{x \to 4} 9 = 9 \]

(h) \[ \lim_{h \to 0} 8 = 8 \]

(i) \[ \lim_{h \to 0} 10 = 10 \]

**EXERCISE 1**

Evaluate:

(a) \[ \lim_{x \to 0} (3 - 2x)^2 \]

(b) \[ \lim_{x \to 4} (x^3 - 1) \]

(c) \[ \lim_{x \to 2} \left( \frac{1}{x + 1} \right)^x \]

(d) \[ \lim_{\theta \to 90^\circ} \cos 2\theta \]

(e) \[ \lim_{x \to 1} \frac{x^2 + 1}{x + 1} \]

(f) \[ \lim_{h \to 0} (2ah + h^2 + a) \]

(g) \[ \lim_{x \to 8} \frac{\sqrt[3]{x^2} + 3\sqrt{x}}{4 - 16x^{-1}} \]

(h) \[ \lim_{x \to 1} 10 \]

(i) \[ \lim_{x \to 4} 2 \]

(j) \[ \lim_{x \to a} t \]

(k) \[ \lim_{h \to 0} 4 \]

(l) \[ \lim_{x \to 2} \frac{x^2 + 3x + 2}{2 + x} \]

(m) \[ \lim_{a \to 3} \frac{a^2 - 9}{a + 3} \]

(n) \[ \lim_{x \to \frac{3}{2}} \frac{4x^2 - 9}{2x + 3} \]

(o) \[ \lim_{x \to \frac{3}{2}} \frac{4x^2 - 1}{1 - 2x} \]

(p) \[ \lim_{x \to \pi} \frac{x^2 - \pi^2}{x - \pi} \]

(q) \[ \lim_{x \to 1} \frac{2x}{x^2 + x} \]

(r) \[ \lim_{x \to 0} \frac{2x}{x^2 + x} \]

(s) \[ \lim_{k \to 0} \frac{1}{k} \left[ 1 - \frac{1}{1 - k} \right] \]

(t) \[ \lim_{x \to 2} \frac{x^3 - 8}{x - 2} \]

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**AVERAGE GRADIENT**

The **average gradient** (or average rate of change) of a function \( f \) between \( x = a \) and \( x = b \) is defined to be the **gradient of the line** joining the points on the graph of the function. We say that the average gradient of \( f \) over the interval \([a; b]\) is the gradient of the line \( AB \).

**EXAMPLE 2**

Find the average gradient of the graph of \( f(x) = x^2 - 4 \) between \( x = -1 \) and \( x = 3 \).

**Solution**

The average gradient of \( f \) between \( x = -1 \) and \( x = 3 \) is the gradient of the line \( AB \) which joins these two points on the graph. We need to determine the \( y \)-coordinates corresponding to the given \( x \)-values. Then it will be easy to determine the gradient of line segment \( AB \). This gradient will be the average gradient of the parabola between the given \( x \)-values.

For \( x = -1 \)
\[
y = (-1)^2 - 4 = -3
\]
\( A(-1; -3) \)

For \( x = 3 \)
\[
y = (3)^2 - 4 = 5
\]
\( B(3; 5) \)

Therefore, the gradient of \( AB \) is:
\[
m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{5 - (-3)}{3 - (-1)} = \frac{8}{4} = 2
\]

Therefore, the **average gradient** of \( f \) between the given \( x \)-values is 2.
EXAMPLE 3

Find the average rate of change of \( g(x) = -x^3 \) over the interval \([-1; 4]\).

**Solution**

For \( x = -1 \)
\[
g(-1) = -(-1)^3 = 1
\]

For \( x = 4 \)
\[
g(4) = -(4)^3 = -64
\]

\[\therefore g(-1) = 1 \quad \therefore g(4) = -64\]

The average rate of change (gradient) of the graph \( y = -x^3 \) between \( x = -1 \) and \( x = 4 \) is given by the gradient of the line segment joining the points \( A(-1;1) \) and \( B(4; -64) \).

\[
m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{-64 - 1}{4 - (-1)} = \frac{-65}{5} = -13
\]

**A general formula for average gradient**

Consider the function \( f \) below.

![Diagram](image)

We can determine a general formula for finding the average gradient of a function \( f \) between two points \( x \) and \( x + h \).

The average gradient of \( f \) is the gradient of the line joining \( A \) and \( B \).

\[
m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{f(x + h) - f(x)}{h}
\]

EXAMPLE 4

Consider the function \( f(x) = 1 - x^2 \)

(a) Determine \( \frac{f(2 + h) - f(2)}{h} \)  
(b) Interpret your answer.
Solutions

(a) \( f(x) = 1 - x^2 \)

\[
\therefore f(2 + h) = 1 - (2 + h)^2
\]

\[
\therefore f(2 + h) = 1 - (4 + 4h + h^2)
\]

\[
\therefore f(2 + h) = 1 - 4 - 4h - h^2
\]

\[
\therefore f(2 + h) = -3 - 4h - h^2
\]

\[
f(2) = 1 - (2)^2 = -3
\]

\[
\therefore f(2) = -3
\]

\[
\therefore \frac{f(2 + h) - f(2)}{h} = \frac{-3 - 4h - h^2 + 3}{h} = \frac{-4h - h^2}{h} = \frac{h(-4 - h)}{h} = -4 - h
\]

(b) This answer represents the average gradient of \( f(x) = 1 - x^2 \) between \( x = 2 \) and \( x = 2 + h \)

EXERCISE 2

1. Consider the function \( f : x \rightarrow x^2 + x \). Determine the average gradient:
   (a) between \( x = 1 \) and \( x = 3 \)
   (b) over the interval \([-3; -2]\)
   (c) between the points \((-1; f(-1))\) and \((4; f(4))\)

2. If \( g(x) = 8 - 2x - x^2 \), determine the average rate of change of \( g \) on the interval \([2; 3]\).

3. Consider the function \( f(x) = -x^2 \)
   (a) Determine \( \frac{f(1 + h) - f(1)}{h} \)
   (b) Interpret your answer graphically.
   (c) Hence find the average gradient of \( f \) on the interval \([1; 4]\)

4. The average rate of change of a function \( f \) between \((3; 4)\) and \((5; a)\) is equal to \( \frac{1}{2} \). Determine the value of \( a \).

GRADIENT FROM FIRST PRINCIPLES

Consider the diagram below. If point B moves closer and closer to A, the line AB will eventually approach the tangent line to \( f \) at A.
The gradient of the tangent line at A is also called the gradient of the graph \( f \) at the point A.

In the diagram which follows, as point B approaches A, the value of \( h \), the distance between the \( x \) values of the points tends towards zero. The gradient of AB will therefore change and approach the gradient of the tangent line at A.

We can represent this scenario in the form of a limit.

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

where \( \frac{f(x + h) - f(x)}{h} \) represents the average gradient of \( f \) between \( x \) and \( x + h \).

We say that: \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \)

represents finding, from FIRST PRINCIPLES:

- the gradient (slope) of the tangent line to \( f \) at any point.
- the gradient (slope) of the function \( f \) at any point.
- the (instantaneous) rate of change of \( f \) at any point.
- the derivative of \( f \) at any point.

The gradient of \( f \) will vary from point to point on \( f \):

If the graph is increasing, the gradient of the tangent is positive (\( f'(x) > 0 \))

If the graph is decreasing, the gradient of the tangent is negative (\( f'(x) < 0 \))

At the turning point, the gradient of the tangent is zero (\( f'(x) = 0 \))
EXAMPLE 5

Find, from first principles, the gradient of \( f(x) = 3x \) at any point.

Solution
We know that the graph of \( y = 3x \) is a straight line with a gradient of 3.
By using first principles, we can verify this fact.

Step 1 Write down the formula for finding gradient from first principles:
\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

Step 2 Write down the given \( f(x) \) and then determine \( f(x+h) \):
\[
f(x) = 3x \\
\therefore f(x+h) = 3(x+h)
\]

Step 3 Substitute the expressions for \( f(x) \) and \( f(x+h) \) into the formula and then simplify the expression and evaluate the limit:
\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]
\[
\therefore f'(x) = \lim_{h \to 0} \frac{3x+3h-3x}{h}
\]
\[
\therefore f'(x) = \lim_{h \to 0} \frac{3h}{h}
\]
\[
\therefore f'(x) = \lim_{h \to 0} 3
\]
\[
\therefore f'(x) = 3
\]

Hence the gradient of the line is 3.

EXAMPLE 6

Find, from first principles, the gradient of \( f(x) = 3 \) at any point.

Solution
We know that the graph of \( y = 3 \) is a horizontal line with a gradient of zero.
By using first principles, we can verify this fact.

Step 1 Write down the formula for finding gradient from first principles:
\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]
Step 2 Write down the given $f(x)$ and then determine $f(x + h)$:

$$f(x) = 3 \quad \therefore f(x + h) = 3$$

Step 3 Substitute the expressions for $f(x)$ and $f(x + h)$ into the formula and then simplify the expression and evaluate the limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$\therefore f'(x) = \lim_{h \to 0} \frac{3 - 3}{h}$$

$$\therefore f'(x) = \lim_{h \to 0} \frac{0}{h}$$

$$\therefore f'(x) = \lim_{h \to 0} 0$$

$$\therefore f'(x) = 0$$

Hence the gradient of the horizontal line is zero.

We can now use this concept of finding the gradient to determine the gradient of other functions at given points.

**EXAMPLE 7**

(a) Find, from first principles, the gradient of $f(x) = 1 - 3x^2$ at any point.

(b) Hence find $f'(-4)$, the derivative of $f$ at $x = -4$.

(c) What is the gradient of the tangent to $f$ at $x = 5$?

**Solution**

(a) **Step 1**

Write down the formula for finding gradient from first principles:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Step 2

Write down the given $f(x)$ and then determine $f(x + h)$:

$$f(x) = 1 - 3x^2 \quad \therefore f(x + h) = 1 - 3(x + h)^2$$

$$\therefore f(x + h) = 1 - 3(x^2 + 2xh + h^2)$$

$$\therefore f(x + h) = 1 - 3x^2 - 6xh - 3h^2$$

Step 3

Substitute the expressions for $f(x)$ and $f(x + h)$ into the formula and then simplify the expression and evaluate the limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$\therefore f'(x) = \lim_{h \to 0} \frac{(1 - 3x^2 - 6xh - 3h^2) - (1 - 3x^2)}{h}$$

$$\therefore f'(x) = \lim_{h \to 0} \frac{1 - 3x^2 - 6xh - 3h^2 - 1 + 3x^2}{h}$$

$$\therefore f'(x) = \lim_{h \to 0} \frac{-6xh - 3h^2}{h}$$

$$\therefore f'(x) = \lim_{h \to 0} \frac{-6x - 3h}{1}$$
\[ f'(x) = \lim_{h \to 0} \frac{h(-6x - 3h)}{h} \]
\[ = \lim_{h \to 0} (-6x - 3h) \]
\[ = -6x - 3(0) \]
\[ = -6x \]

(b) Since \( f'(x) = -6x \) represents the gradient of the graph at any point on the graph, it is now easy to determine the gradient (derivative) at \( x = -4 \):
\[ f'(-4) = -6(-4) = 24 \]

(c) The gradient of the tangent to the graph at \( x = 5 \) can now also be determined:
\[ f'(5) = -6(5) = -30 \]

**EXAMPLE 8**

Determine, from the first principles, the derivative of \( f(x) = \frac{1}{x} \).

**Solution**

**Step 1** Write down the formula for finding gradient from first principles:
\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]

**Step 2** Write down the given \( f(x) \) and then determine \( f(x + h) \):
\[ f(x) = \frac{1}{x} \quad \therefore f(x + h) = \frac{1}{x + h} \]

**Step 3** Substitute the expressions for \( f(x) \) and \( f(x + h) \) into the formula and then simplify the expression and evaluate the limit:
\[ f'(x) = \lim_{h \to 0} \frac{1}{h} \left( \frac{x + h}{x} - \frac{x}{x} \right) \]
\[ = \lim_{h \to 0} \frac{x + h}{x} - 1 \]
\[ = \lim_{h \to 0} \frac{x + h - x}{x} \]
\[ = \lim_{h \to 0} \frac{h}{x} \] (LCD of numerator is \( x(x + h) \))
\[ = \lim_{h \to 0} \frac{x + h}{x} \]
\[ = \lim_{h \to 0} \frac{-h}{x(x + h)} \]
\[ = \lim_{h \to 0} \frac{-1}{x(x + h)} \]
\[ = \frac{-1}{x(x + 0)} = -\frac{1}{x^2} \]
EXERCISE 3

1. Determine the derivatives of the following at any point by using first principles.
   (a) \( f(x) = x \)  
   (b) \( f(x) = 2x \)  
   (c) \( g(x) = -3x \)  
   (d) \( f(x) = 4 \)  
   (e) \( f(x) = 5 \)  
   (f) \( g(x) = -7 \)  
   (g) \( f(x) = x^2 \)  
   (h) \( f(x) = 2x^2 \)  
   (i) \( g(x) = -4x^2 \)  
   (j) \( f(x) = 1 + x^2 \)  
   (k) \( f(x) = 3 - x^2 \)  
   (l) \( g(x) = x^3 \)  
   (m) \( f(x) = \frac{2}{x} \)  
   (n) \( g(x) = -\frac{3}{x} \)  

2. Consider \( f : x \to x^2 - x \)
   (a) Find the average rate of change of \( f \) over the interval \([1; 2]\).
   (b) Find the instantaneous rate of change of \( f \) at \( x = 1 \) by using first principles.
   (c) Determine \( \lim_{h \to 0} \frac{f(x+h) - f(3)}{h} \) and interpret your answer.

3. Consider the function \( f(x) = -2ax^2 + 2x \)
   (a) Determine \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \)
   (b) Determine:
      (1) \( f(3) \)  
      (2) \( f'(3) \)  
      (3) the gradient of the tangent to \( f \) at \( x = -4 \)  
      (4) the rate of change of \( f \) at \( x = 2 \)

THE RULES OF DIFFERENTIATION

You will probably agree that the method of finding gradients from first principles is quite tedious, especially if the functions are complicated. We will now investigate other rules for finding gradients. They are called the rules of differentiation and are much easier to use. In an examination, you will only be required to determine the gradient of one simple function by using first principles. Otherwise, the rules of differentiation are used in all other situations.

Investigation

In the tables below, the gradients of a number of functions have been recorded. You can use first principles to verify the gradients of a few of the functions.

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 )</td>
<td>( 2x )</td>
</tr>
<tr>
<td>( x^3 )</td>
<td>( 3x^2 )</td>
</tr>
<tr>
<td>( x^4 )</td>
<td>( 4x^3 )</td>
</tr>
<tr>
<td>( x^5 )</td>
<td>( 5x^4 )</td>
</tr>
<tr>
<td>( x^6 )</td>
<td>( 6x^5 )</td>
</tr>
<tr>
<td>( x^{100} )</td>
<td>( 100x^{99} )</td>
</tr>
<tr>
<td>( 3x^{1000} )</td>
<td>( 3\times1000x^{999} )</td>
</tr>
<tr>
<td>( kx^n )</td>
<td>( k\times nx^{n-1} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( 2x )</td>
<td>( 2 )</td>
</tr>
<tr>
<td>( 3x )</td>
<td>( 3 )</td>
</tr>
<tr>
<td>( 5x )</td>
<td>( 5 )</td>
</tr>
<tr>
<td>( kx )</td>
<td>( k )</td>
</tr>
<tr>
<td>( 5 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( 7 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( k )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>
Discussion

What can you conclude, in general, about the derivative of the functions?

\[ f(x) = kx^n, \quad f(x) = kx \quad \text{and} \quad f(x) = k \quad \text{where} \quad k \quad \text{is a constant?} \]

Solution

The gradient of \( f(x) = kx^n \) is \( f'(x) = k \times nx^{n-1} \) (This is called the power rule)

The gradient of \( f(x) = kx \) where \( k \) is a constant is \( f'(x) = k \)

The graph of \( y = kx \) is a linear function and the coefficient of \( x \) is the gradient.

The gradient of \( f(x) = k \) where \( k \) is a constant is \( f'(x) = 0 \)

The graph of \( y = k \) is a horizontal line where the gradient is 0.

The following rules of differentiation are applicable for determining the gradient of a function given that the value of \( k \) is a constant.

**Rule 1**

If \( f(x) = kx^n \), then \( f'(x) = k \times nx^{n-1} \)

**Example:**

If \( f(x) = 8x^3 \), then \( f'(x) = 8 \times 3x^{3-1} = 24x^2 \)

**Rule 2**

If \( f(x) = kx \), then \( f'(x) = k \)

**Example:**

If \( f(x) = 9x \), then \( f'(x) = 9 \)

**Rule 3**

If \( f(x) = k \), then \( f'(x) = 0 \)

**Example:**

If \( f(x) = 9 \), then \( f'(x) = 0 \)

If \( f(x) = p \) (where \( p \) is a constant), then \( f'(x) = 0 \)

Exponent and surd definitions which are required for doing differentiation

Before discussing the examples that follow, it is critical that you go over the exponent and surd laws from Grade 10 and 11. The following definitions will also prove helpful to you in this section of calculus.

\[
\begin{align*}
(a) \quad \frac{x^n}{a} &= \frac{1}{a} x^n \\
(b) \quad \frac{a}{x^n} &= ax^{-n} \\
(c) \quad \frac{a}{bx^n} &= \frac{a}{b} x^{-n} \\
(d) \quad \frac{a}{(bx)^n} &= a (bx)^{-n} \\
(e) \quad \frac{a}{\sqrt[n]{x^m}} &= a \cdot x^{\frac{-m}{n}} \\
(f) \quad \sqrt[n]{ax^m} &= \sqrt[n]{a} \cdot x^{\frac{m}{n}}
\end{align*}
\]

There are different symbols which are used to represent gradient. We will now discuss some basic examples to illustrate the rules of differentiation as well as the different symbols used. The following symbols all mean the same thing (gradient, derivative, slope, rate of change):

\[ f'(x) \quad \frac{dy}{dx} \quad D_x \]
EXAMPLE 9

Determine:

<table>
<thead>
<tr>
<th></th>
<th>Rule 1:</th>
<th>Rule 1:</th>
<th>Rule 1:</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>[ f'(x) \text{ if } f(x) = 3x^6 ]</td>
<td>[ y = 3x^6 ]</td>
<td>[ D_x[3x^6] ]</td>
</tr>
<tr>
<td>(a)</td>
<td>[ f(x) = 3x^6 ]</td>
<td>[ \frac{dy}{dx} = 3 \cdot 6x^6-1 ]</td>
<td>[ = 3 \cdot 6x^6-1 ]</td>
</tr>
<tr>
<td></td>
<td>[ \therefore f'(x) = 3x^5 ]</td>
<td>[ \frac{dy}{dx} = 18x^5 ]</td>
<td>[ = 18x^5 ]</td>
</tr>
<tr>
<td>(b)</td>
<td>[ f'(x) \text{ if } f(x) = 10x ]</td>
<td>[ y = 10x ]</td>
<td>[ D_x[10x] ]</td>
</tr>
<tr>
<td>(b)</td>
<td>[ f(x) = 10x ]</td>
<td>[ \frac{dy}{dx} = 10 ]</td>
<td>[ = 10 ]</td>
</tr>
<tr>
<td>(c)</td>
<td>[ f'(x) \text{ if } f(x) = -8 ]</td>
<td>[ y = -8 ]</td>
<td>[ D_x[-8] ]</td>
</tr>
<tr>
<td>(c)</td>
<td>[ f(x) = -8 ]</td>
<td>[ \frac{dy}{dx} = 0 ]</td>
<td>[ = 0 ]</td>
</tr>
<tr>
<td>(d)</td>
<td>[ f'(x) \text{ if } f(x) = m ] (m is a constant)</td>
<td>[ y = m ]</td>
<td>[ D_x[m] ]</td>
</tr>
<tr>
<td>(d)</td>
<td>[ f(x) = m ]</td>
<td>[ \frac{dy}{dx} = 0 ]</td>
<td>[ = 0 ]</td>
</tr>
<tr>
<td>(e)</td>
<td>[ g'(x) \text{ if } g(x) = \frac{x^3}{3} ]</td>
<td>[ y = \frac{x^3}{3} ]</td>
<td>[ D_x\left[\frac{x^3}{3}\right] ]</td>
</tr>
<tr>
<td>(e)</td>
<td>[ g(x) = \frac{x^3}{3} ]</td>
<td>[ \frac{dy}{dx} = \frac{1}{3} \times 3x^{3-1} ]</td>
<td>[ = \frac{1}{3} \times 3x^{3-1} ]</td>
</tr>
<tr>
<td></td>
<td>[ \therefore g'(x) = \frac{1}{3} \times 3x^{3-1} ]</td>
<td>[ \frac{dy}{dx} = x^2 ]</td>
<td>[ \text{Rule 1} ]</td>
</tr>
<tr>
<td></td>
<td>[ \therefore g'(x) = x^2 ]</td>
<td>[ \frac{dy}{dx} = x^2 ]</td>
<td>[ = x^2 ]</td>
</tr>
</tbody>
</table>

First use exp def (a):

\[ g(x) = \frac{x^3}{3} \]

Now use Rule 1:

\[ \therefore g'(x) = \frac{1}{3} \times 3x^{3-1} \]

\[ \therefore g'(x) = x^2 \]
<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>(f)</td>
<td>$p'(x)$ if $p(x) = \frac{3}{2x^4}$</td>
<td>(f)</td>
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<tr>
<td></td>
<td>$p(x) = \frac{3}{2x^4}$</td>
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<tr>
<td></td>
<td>First use exp def (c):</td>
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<tr>
<td></td>
<td>$\therefore p(x) = \frac{3}{2}x^{-4}$</td>
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<tr>
<td></td>
<td>Now use Rule 1:</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\therefore p'(x) = \frac{3}{2}x^{-4-1}$</td>
<td></td>
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<td></td>
<td>$\therefore p'(x) = -6x^{-5}$</td>
<td></td>
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<td></td>
<td>$\therefore p'(x) = -6$</td>
<td></td>
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<tr>
<td>(g)</td>
<td>$f'(t)$ if $f(t) = \frac{\pi}{2}t^8$</td>
<td>(g)</td>
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<tr>
<td></td>
<td>$f(t) = \frac{\pi}{2}t^8$</td>
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<tr>
<td></td>
<td>Rule 1:</td>
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<tr>
<td></td>
<td>$\therefore f'(t) = \frac{\pi}{2} \times 8t^7$</td>
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<td></td>
<td>$\therefore f'(t) = 4\pi t^7$</td>
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<tr>
<td>(h)</td>
<td>$f'(x)$ if $f(x) = t^2 + x$</td>
<td>(h)</td>
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<tr>
<td></td>
<td>$f(x) = t^2 + 1x$</td>
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<td></td>
<td>$\therefore f'(x) = 0 + 1$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\therefore f'(x) = 1$</td>
<td></td>
</tr>
<tr>
<td>(i)</td>
<td>$f'(x)$ if $f(x) = \sqrt{x}$</td>
<td>(i)</td>
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<tr>
<td></td>
<td>$f(x) = x^{\frac{1}{2}}$</td>
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<tr>
<td></td>
<td>$\therefore f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$</td>
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<tr>
<td></td>
<td>$\therefore f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\therefore f'(x) = \frac{1}{2} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}}$</td>
<td></td>
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</tbody>
</table>
EXAMPLE 10

(a) If \( f(x) = x^4 + 6x + 7 \), determine \( f'(2) \), the derivative of \( f \).

\[
f(x) = x^4 + 6x + 7
\]
\[
\therefore f'(x) = 4x^3 + 6 + 0 \quad \text{(Rule 1, 2 and 3)}
\]
\[
\therefore f'(x) = 4x^3 + 6
\]
\[
\therefore f'(2) = 4(2)^3 + 6
\]
\[
\therefore f'(2) = 38
\]

(b) Determine \( D_x \left[ px^2 + p^2 + \frac{1}{x^2} \right] \) where \( p \) is a constant

\[
D_x \left[ px^2 + p^2 + \frac{1}{x^2} \right]
\]
\[
= D_x \left[ px^2 + p^2 + x^{-2} \right]
\]
\[
= p \times 2x^{2-1} + 0 + (-2)x^{-2-1}
\]
\[
= 2px - 2x^{-3}
\]
\[
= 2px - \frac{2}{x^3}
\]

Note: Be aware of the placement of the equal signs and the use of the symbols in the previous examples.

More advanced types of differentiation

Remember the following points when differentiating:

Make sure that you first remove any:
- brackets
- divide lines
- root signs

Always get the terms in a form which allows you to apply the rules of differentiation to each.

EXAMPLE 11

Determine the following:

(a) \( f'(x) \) if \( f(x) = (4x - 3)^2 \)

Here we first need to expand and simplify so as to remove the brackets.

\[
f(x) = (4x - 3)^2
\]
\[
\therefore f(x) = 16x^2 - 24x + 9
\]
\[
\therefore f'(x) = 16 \times 2x^{2-1} - 24 + 0
\]
\[
\therefore f'(x) = 32x - 24
\]
(b) \( \frac{dy}{dx} \) if \( y = \frac{1}{\sqrt[3]{x^3}} \)

Here we need to first get rid of the divide line over the variable as well as the root sign.

\[ y = \frac{1}{\sqrt[3]{x^3}} \]

\[ \therefore y = \frac{1}{x^\frac{3}{4}} \] \((\sqrt[3]{a^m} = a^{\frac{m}{3}})\)

\[ \therefore y = \frac{1}{2} x^{-\frac{3}{4}} \] \(\text{(Exponential definition (c))}\)

\[ \therefore \frac{dy}{dx} = \frac{1}{2} x^{-\frac{3}{4}} \times -\frac{3}{4} x^{-\frac{3}{4}-1} \] \(\text{(Rule 1)}\)

\[ \therefore \frac{dy}{dx} = \frac{3}{8} x^{-\frac{7}{4}} \]

\[ \frac{dy}{dx} = -\frac{3}{8x^\frac{7}{4}} \]

\[ \therefore \frac{dy}{dx} = -\frac{3}{8\sqrt[4]{x^7}} \]

(c) \( D_x \left[ \frac{x^2 - x - 6}{2x^2} \right] \)

= \( D_x \left[ \frac{x^2}{2x^2} - \frac{x}{2x^2} - \frac{6}{2x^2} \right] \)

= \( D_x \left[ \frac{1}{2} - \frac{1}{2x} - \frac{3}{x^2} \right] \)

= \( D_x \left[ \frac{1}{2} - \frac{1}{2} x^{-1} - 3x^{-2} \right] \)

= \( 0 - \frac{1}{2}(-1)x^{-1-1} - 3(-2)x^{-2-1} \)

= \( \frac{1}{2} x^{-2} + 6x^{-3} \)

= \( \frac{1}{2x^2} + \frac{6}{x^3} \)

(d) \( D_x \left[ \frac{3x^3 - 7x^2 - 6x}{3 - x} \right] \)

= \( D_x \left[ \frac{x(3x^2 - 7x - 6)}{3 - x} \right] \)

= \( D_x \left[ \frac{x(3x + 2)(x - 3)}{3 - x} \right] \)

= \( D_x \left[ \frac{x(3x + 2)(x - 3)}{-x + 3} \right] \)

= \( D_x \left[ \frac{3x^2 + 2}{-x + 3} \right] \)

= \( D_x \left[ -3x^2 - 2x \right] \)

= \( -6x - 2 \)
EXERCISE 4

1. Determine the following:
   
   (a) \(f'(x)\) if \(f(x) = x^6\)
   
   (b) \(\frac{dy}{dx}\) if \(y = 8x^4\)
   
   (c) \(D_x[-7x^5]\)
   
   (d) \(f'(x)\) if \(f(x) = \frac{1}{4}x^8\)
   
   (e) \(\frac{dy}{dx}\) if \(y = 9x\)
   
   (f) \(D_x[-7x]\)
   
   (g) \(g'(x)\) if \(g(x) = 15\)
   
   (h) \(\frac{dy}{dx}\) if \(y = px\)
   
   (i) \(D_x\left[\frac{1}{4}x\right]\)
   
   (j) \(g'(x)\) if \(g(x) = \frac{1}{5}\)
   
   (k) \(\frac{dy}{dx}\) if \(y = \frac{x}{5}\)
   
   (l) \(D_x\left[\frac{x^2}{5}\right]\)
   
   (m) \(p'(x)\) if \(p(x) = \frac{5}{x^2}\)
   
   (n) \(\frac{dy}{dx}\) if \(y = \frac{4}{5x}\)
   
   (o) \(D_x\left[\frac{3x^2}{2} + \frac{3x}{2}\right]\)
   
   (p) \(p'(x)\) if \(p(x) = \frac{5}{4}x^4 + \frac{5}{4}\)
   
   (q) \(\frac{dy}{dx}\) if \(y = \frac{4}{5x^5} + t^2\) (\(t\) is a constant)
   
   (r) \(D_x[\frac{1}{3}x^3 - 15x^2 + 2x + 3]\)
   
   (s) \(f'(x)\) if \(f(x) = 1 - x + 3x^4\)
   
   (t) \(D_x\left[\frac{1}{3}x^6 - \frac{3}{4}x^8 + \frac{1}{2}x^2\right]\)
   
   (u) \(D_x\left[a^2x^3 + a^2 + x^3\right]\) (\(a\) is a constant)
   
   (v) \(D_x\left[\pi^3x + \pi^3x + \pi^4 + x^0\right]\)

2. Determine:

   (a) \(D_x[(2x - 3)(x + 4)]\)
   
   (b) \(D_x\left[\frac{2}{3x^4}\right]\)
   
   (c) \(D_x\left[\sqrt[3]{x} + \frac{1}{\sqrt{x}}\right]\)
   
   (d) \(D_x[2x(x^2/3 - 3) + 6ax]\)
   
   (e) \(D_x[8\sqrt{x} - \frac{1}{2x} + 3m^4]\)
   
   (f) \(D_x[\sqrt{5x} + \sqrt{5} \cdot x + \sqrt{5}]\)
   
   (g) \(D_x[(x^2 - \sqrt{x})^2]\)
   
   (h) \(D_t\left[\frac{t^3 - t^2 + 1}{t}\right]\)
   
   (i) \(D_x\left[\frac{3\sqrt{x^2 - x^2}}{\sqrt{x}}\right]\)
   
   (j) \(D_m\left[\frac{(1-m)^2}{2m}\right]\)
   
   (k) \(D_p\left[\frac{2p^2 + 3p + 1}{p + 1}\right]\)
   
   (l) \(D_x\left[\frac{x^2 - 9}{3 - x}\right]\)
   
   (m) \(D_x\left[\frac{x^3 - 2x^2 - x + 2}{x + 1}\right]\)
   
   (n) \(D_x\left[\left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right)^2\right]\)
(o) \[ D_x \left[ 3x^2 + (3x)^2 + 3x^0 \right] \]

(p) \[ D_x \left[ \frac{2x^3 + 2x^2 - 24x}{3 - x} \right] \]

(q) \[ \frac{dy}{dx} \text{ if } y = \left( 2\sqrt{x} - \frac{1}{x} \right)^2 \]

(r) \[ D_x \left[ \frac{1}{\sqrt{x}} \left( x^3 - 2x^2 + 3x \right) \right] \]

(s) \[ \frac{dy}{dx} \text{ if } y = (1 - x^2)(1 - 2x^{1/2}) \]

(t) \[ \frac{dy}{dx} \text{ if } y = \frac{2x^2 - \sqrt{x} + 5}{\sqrt{x}} \]

(u) \[ \frac{dy}{dt} \text{ if } y = (t^{\frac{1}{4}} - t^{\frac{1}{3}})^2 \]

(v) \[ \frac{dx}{dy} \text{ if } x = 4^{-1} y^4 - (y^{-1} - y)^2 \]

(w) \[ \frac{dy}{dx} \text{ at } x = 1 \text{ if } y = (2x^2) \left( \frac{9-x}{x^3} \right) \]

(x) \[ \frac{d}{dx} \left[ \left( x^3 + \frac{1}{2x^3} \right)^{\frac{1}{2}} \right] \]

(y) \[ f'(x) \text{ if } f(x) = (x^2 - 2\sqrt{x})^2 \]

(z) \[ \frac{ds}{dt} \text{ if } s = \frac{3\sqrt{t^5} + 2\sqrt{t^3}}{3\sqrt{t^5}} \]

(aa) \[ D_x \left[ \sqrt[3]{x} - \frac{x^2 - 2x}{x^2} \right] \]

(bb) \[ D_x \left[ \frac{x^{-1} + 2}{\sqrt{x^3 + 2x^4}} \right] \]

3. (a) Find \( \frac{dy}{dx} \) if \( \frac{y}{2x} = (1-x)^2 \)

(b) Find \( \frac{dy}{dx} \) if \( xy + y = x^2 - 1 \)

(c) If \( 4t^2 - 2tr + r - 1 = 0 \), show that \( \frac{dr}{dt} = 2 \)

(d) Determine \( \frac{dy}{dx} \) if \( 2(y + 6) - 12x^2 = 8x(x-4)(x-1) \)

(e) If \( f(x) = \frac{16}{x} \), determine \( f'(4x) - 4f'(x) \)

4. (a) If \( y = 2x(3-x) \) and \( z = \frac{y}{2} \), determine:

(1) \( \frac{dy}{dx} \)  \hspace{1cm} (2) \( \frac{dz}{dy} \)  \hspace{1cm} (3) \( \frac{dz}{dx} \)

(b) If \( \sqrt{x} = y^\frac{1}{3} \) and \( y = z^{-3} \), determine:

(1) \( \frac{dy}{dx} \)  \hspace{1cm} (2) \( \frac{dx}{dz} \)  \hspace{1cm} (3) \( \frac{dy}{dz} \)
GRAPHICAL APPLICATIONS OF CALCULUS

A central principle of calculus is that at any point \( x = a \) on the graph of a function \( f \), the gradient of \( f \) at \( x = a \) is the same as the gradient of the tangent to \( f \) at \( x = a \). In other words, \( m_t = f'(a) \).

**Increasing and decreasing functions**

A function \( f \) strictly increases over an interval \((a ; b)\) if \( f'(x) > 0 \) for all \( x \in (a ; b) \). The gradient of the tangent to \( f \) at any point in this interval will be positive since the tangent line slopes to the right.

A function \( f \) strictly decreases over an interval \((a ; b)\) if \( f'(x) < 0 \) for all \( x \in (a ; b) \). The gradient of the tangent to \( f \) at any point in this interval will be negative since the tangent line slopes to the left.

**Stationary points**

At any stationary point on a graph \( f \), the gradient is zero. If \( x = p \) is a stationary point then \( f'(p) = 0 \) since the gradient of the tangent line at \( x = p \) is zero (the line is horizontal). At a stationary point, the function neither increases nor decreases. Turning points are stationary points since at a turning point \( f'(x) = 0 \).

At \( x = a \) there is a maximum turning point.
For all values of \( x < a \), the graph is increasing and \( f'(x) > 0 \)
For all values of \( x > a \), the graph is decreasing and \( f'(x) < 0 \)

At \( x = b \) there is a minimum turning point.
For all values of \( x < b \), the graph is decreasing and \( f'(x) < 0 \)
For all values of \( x > b \), the graph is increasing and \( f'(x) > 0 \)
EXAMPLE 12

Consider the graph of \( f(x) = x^2 + 2x - 3 \)

(a) Determine the coordinates of the turning point.

(b) Determine the interval over which \( f \) is increasing and decreasing.

(c) Determine the gradient of the tangent to \( f \) at \( x = -4 \).

(d) Determine the coordinates of the point on \( f \) for which the gradient is equal to 12.

Solutions

(a) There are actually four methods available to determine the turning point.

Method 1 (Complete the square)

\[
y = x^2 + 2x - 3
\]

\[
\therefore y = x^2 + 2x + \left[ \frac{1}{2} (2) \right]^2 - \left[ \frac{1}{2} (2) \right]^2 - 3
\]

\[
\therefore y = x^2 + 2x + 1 - 1 - 3
\]

\[
\therefore y = (x + 1)^2 - 4
\]

Axis of symmetry:

\[
x + 1 = 0
\]

\[
\therefore x = -1
\]

y-value of turning point: -4

\[
\therefore \text{The turning point is } (-1; -4)
\]

Method 2 (The formula)

\[
y = 1x^2 + 2x - 3
\]

\[
a = 1 \quad b = 2
\]

\[
x = -\frac{b}{2a}
\]

\[
\therefore x = -\frac{2}{2(1)}
\]

\[
\therefore x = -1
\]

Substitute \( x = -1 \) into the original equation to get the corresponding y-value.

\[
y = (-1)^2 + 2(-1) - 3
\]

\[
\therefore y = 1 - 2 - 3
\]

\[
\therefore y = -4
\]

\[
\therefore \text{The turning point is } (-1; -4)
\]

Method 3 (Differentiation)

At a turning point

\[
f'(x) = 2x + 2
\]

\[
f'(-1) = 0 \therefore 0 = 2x + 2
\]

\[
\therefore -2x = 2
\]

\[
\therefore x = -1
\]

To get the corresponding value of y, determine \( f(-1) \)

\[
f(-1) = (-1)^2 + 2(-1) - 3
\]

\[
\therefore f(-1) = 1 - 2 - 3
\]

\[
\therefore f(-1) = -4
\]

\[
\therefore \text{The turning point is } (-1; -4)
\]

Method 4 (Symmetry)

\[
x = \frac{\text{sum of } x\text{-intercepts}}{2}
\]

\[
\therefore x = \frac{-3 + 1}{2}
\]

\[
\therefore x = -1
\]

Substitute \( x = -1 \) into the original equation to get the corresponding y-value.

\[
y = (-1)^2 + 2(-1) - 3
\]

\[
\therefore y = 1 - 2 - 3
\]

\[
\therefore y = -4
\]
(b) The graph of $f$ increases for $x > -1$
   The graph of $f$ decreases for $x < -1$

(c) The gradient of the tangent to $f$ at $x = -4$ is given by $f'(-4)$

   \[ f(x) = x^2 + 2x - 3 \]
   \[ \therefore f'(x) = 2x + 2 \]
   \[ \therefore f'(-4) = 2(-4) + 2 \]
   \[ \therefore f'(-4) = -6 \]

(d) The gradient at any point is given by $f'(x)$. If the gradient is 12 then we can state that:
   \[ f'(x) = 12 \]
   \[ \therefore 2x + 2 = 12 \]
   \[ \therefore 2x = 10 \]
   \[ \therefore x = 5 \]
   Now determine the corresponding $y$-value:
   \[ f(5) = (5)^2 + 2(5) - 3 \]
   \[ \therefore f(5) = 25 + 10 - 3 \]
   \[ \therefore f(5) = 32 \]
   Therefore, the coordinates of the point on the graph where the gradient is 12 is $(5; 32)$.

**EXERCISE 5**

1. Consider $f : x \rightarrow x^2 - 2x + 8$.
   (a) Find the coordinates of the roots and $y$-intercept.
   (b) Find the coordinates of the turning point.
   (c) Find the intervals over which $f$ is increasing and decreasing.
   (d) Sketch the graph of $f$.
   (e) Determine the gradient of the tangent to $f$ at $x = 3$.
   (f) Determine the point on $f$ for which the gradient of $f$ is 4.

2. Consider $f : x \rightarrow x^2 - x - 6$.
   (a) Find the coordinates of the roots and $y$-intercept.
   (b) Find the coordinates of the turning point.
   (c) Sketch the graph of $f$.
   (d) Determine the derivative at $x = -1$.
   (e) Find the point on $f$ for which the gradient of the tangent to $f$ is 7.

3. Consider the function $g(x) = ax^2 + bx + c$. Prove that the $x$-coordinate of the turning point for the function $g$ is given by $x = -\frac{b}{2a}$ where $a \neq 0$. 

THE GRAPH OF A CUBIC FUNCTION

The cubic function has the general equation \( f(x) = ax^3 + bx^2 + cx + d \).
The graph of a cubic function can take on one of the following shapes depending on whether the value of \( a \) is positive or negative.

The graph of a cubic function has two stationary points called turning points (local maximum and minimum) as well as what is called a point of inflection. Let’s explore this further.

Turning points
Point of inflection

A point of inflection on the graph of a cubic function is the point at which the **concavity** of the function changes. Cubic functions can change from being concave down (sad or convex) to concave up (happy) or from concave up to concave down. An important principle is that at a point of inflection, the second derivative is equal to zero, i.e. \( f''(x) = 0 \). The reason for this will be investigated later on in this chapter.

There are some cubic functions which can have a stationary point that is also a point of inflection.

\[ a > 0 \]

\[ a < 0 \]
We are now in a position to draw the graphs of cubic functions. These functions have stationary points as well as points of inflection.

### Rules for sketching the graphs of cubic functions

#### Intercepts with the axes:
- For the y-intercept, let \( x = 0 \) and solve for \( y \).
- For the x-intercepts, let \( y = 0 \) and solve for \( x \).

#### Stationary points:
- Determine \( f'(x) \), equate it to zero and solve for \( x \).
- Substitute the \( x \)-values of the stationary points into the original equation to obtain the corresponding \( y \)-values.
- If the function has **two stationary points**, establish whether they are maximum or minimum turning points by referring to the shape \( (a > 0 \text{ or } a < 0) \).

#### Points of inflection:
- If the cubic function has only **one stationary point**, this point will be a point of inflection that is also a stationary point. Refer to the shape to see what kind of point of inflection it is \( (a > 0 \text{ or } a < 0) \).
- For points of inflection that are not stationary points, find the second derivative \( f''(x) \), equate it to 0 and solve for \( x \).

---

### EXAMPLE 13

Sketch the graph of the function \( f(x) = x^3 - 6x^2 + 9x \).

#### Intercepts with the axes:
- **y-intercept:** Let \( x = 0 \)
  \[
  f(0) = (0)^3 - 6(0)^2 + 9(0) = 0
  \]
- **x-intercepts:** Let \( y = 0 \)
  \[
  0 = x^3 - 6x^2 + 9x
  \]

#### Stationary points:
- \( f(x) = x^3 - 6x^2 + 9x \)
- \( f'(x) = 3x^2 - 12x + 9 \)
- \( \therefore f'(x) = 3(x^2 - 4x + 3) \)
- \( \therefore 0 = (x - 3)(x - 1) \)
- \( \therefore x = 3 \text{ or } x = 1 \)

Now substitute the values of \( x \) into the original equation to get the corresponding values of \( y \).

For \( x = 3 \):
- \( f(3) = (3)^3 - 6(3)^2 + 9(3) \)
- \( \therefore f(3) = 27 - 54 + 27 \)
- \( \therefore f(3) = 0 \)
Therefore there is a stationary point at \((3; 0)\)

For \( x = 1 \):
- \( f(1) = (1)^3 - 6(1)^2 + 9(1) \)
- \( \therefore f(1) = 1 - 6 + 9 \)
- \( \therefore f(1) = 4 \)
Therefore there is a stationary point at \((1; 4)\)
We now refer to the rules of shape to determine what the graph looks like.
Since \( a = 1 > 0 \) in the original equation \( f(x) = x^3 - 6x^2 + 9x \) and given the fact that there are two stationary points, it is clear that the graph has the following shape:

This means that \((1; 4)\) is a local maximum turning point and \((3; 0)\) is a local minimum turning point.

**Point of inflection:**
\[
f'(x) = 3x^2 - 12x + 9
\]
\[
\therefore f''(x) = 6x - 12
\]
\[
\therefore 0 = 6x - 12
\]
\[
\therefore -6x = -12
\]
\[
\therefore x = 2
\]
\[
\therefore f(2) = (2)^3 - 6(2)^2 + 9(2)
\]
\[
\therefore f(2) = 2
\]

Therefore there is a point of inflection at \((2; 2)\)

Now plot all the above points on the following set of axes and then draw a graph of the cubic function.
EXAMPLE 14

Sketch the graph of the function \( f(x) = -x^3 - 4x^2 + 3x + 18 \).

Intercepts with the axes:

**y-intercept:** Let \( x = 0 \)

\[ f(0) = -(0)^3 - 4(0)^2 + 3(0) + 18 = 18 \]

**x-intercepts:** Let \( y = 0 \)

\[ 0 = -x^3 - 4x^2 + 3x + 18 \]
\[ \therefore x^3 + 4x^2 - 3x - 18 = 0 \]
\[ \therefore (x - 2)(x^2 + 6x + 9) = 0 \quad \text{(by inspection)} \]
\[ \therefore (x - 2)(x + 3)^2 = 0 \]
\[ \therefore x = 2 \quad \text{or} \quad x = -3 \]

Stationary points:

\[ f(x) = -x^3 - 4x^2 + 3x + 18 \]
\[ \therefore f'(x) = -3x^2 - 8x + 3 \]
\[ \therefore 0 = -3x^2 - 8x + 3 \]
\[ \therefore 3x^2 + 8x - 3 = 0 \]
\[ \therefore (3x - 1)(x + 3) = 0 \]
\[ \therefore x = \frac{1}{3} \quad \text{or} \quad x = -3 \]

\[ f\left(\frac{1}{3}\right) = -\left(\frac{1}{3}\right)^3 - 4\left(\frac{1}{3}\right)^2 + 3\left(\frac{1}{3}\right) + 18 \]
\[ \therefore f\left(\frac{1}{3}\right) = \frac{1}{27} - \frac{4}{9} + 1 + 18 \]
\[ \therefore f\left(\frac{1}{3}\right) = \frac{14}{27} \]

\[ f(-3) = -(-3)^3 - 4(-3)^2 + 3(-3) + 18 \]
\[ \therefore f(-3) = -(-27) - 4(9) - 9 + 18 \]
\[ \therefore f(-3) = 27 - 36 - 9 + 18 \]
\[ \therefore f(-3) = 0 \]

There are stationary points at \( \left(\frac{1}{3}; 18 \frac{14}{27}\right) \) and \( (-3; 0) \).

We now refer to the rules of shape to determine what the graph looks like.

Since \( a = -1 < 0 \) in the original equation \( f(x) = -x^3 - 4x^2 + 3x + 18 \) and given the fact that there are two stationary points, it is clear that the graph has the following shape:
This means that \((-3; 0)\) is a local minimum turning point and \(\left(\frac{1}{3}; 18 \frac{14}{27}\right)\) is a local maximum turning point.

**Point of inflection:**

\[
f'(x) = -3x^2 - 8x + 3
\]
\[
\therefore f''(x) = -6x - 8
\]
\[
\therefore 0 = -6x - 8
\]
\[
\therefore 6x = -8
\]
\[
\therefore x = -\frac{8}{6} = -\frac{4}{3}
\]
\[
\therefore f\left(-\frac{4}{3}\right) = -\left(-\frac{4}{3}\right)^3 - 4\left(-\frac{4}{3}\right)^2 + 3\left(-\frac{4}{3}\right) + 18
\]
\[
\therefore f\left(-\frac{4}{3}\right) = 9 \frac{7}{27}
\]

Point of inflection at \(\left(-1\frac{1}{3}; 9 \frac{7}{27}\right)\)

Now plot all the above points on the following set of axes and then draw a graph of the cubic function.

**EXAMPLE 15**

Sketch the graph of \(g(x) = -x^3 - 1\)

**Intercepts with the axes:**

- **y-intercept:** Let \(x = 0\)
  \[g(0) = -(0)^3 - 1 = -1\]
- **x-intercept:** Let \(y = 0\)
  \[0 = -x^3 - 1\]
  \[\therefore x^3 = -1\]
  \[\therefore x = -1\]

**Stationary points:**

- **Critical point:**
  \[g(x) = -x^3 - 1\]
  \[g(0) = -(0)^3 - 1 = -1\]
  \[\therefore g'(x) = -3x^2\]
  \[\therefore 0 = -3x^2\]
  \[\therefore 3x^2 = 0\]
  \[\therefore x^2 = 0\]
  \[\therefore x = 0\]
There is one stationary point at \((0 ; -1)\).
This point is therefore also a point of inflection.

We now refer to the rules of shape to determine what the graph looks like.

Since \(a = -1 < 0\) in the original equation \(g(x) = -1x^3 - 1\)
and since the function only has one stationary point
which is also a point of inflection, the graph will look like:

Now plot all the above points on the following set of axes and then draw a graph of
the cubic function.

EXERCISE 6

1. For each of the following functions given below:
   (1) Determine the intercepts with the axes.
   (2) Determine the coordinates of the stationary points and establish whether
       they are maximum or minimum turning points, or points of inflection.
   (3) Determine the coordinates of the point of inflection which is not a
       stationary point.
   (4) Now sketch the graph of the function on a set of axes. Write down the
       values of \(x\) for which the function increases and/or decreases.

   (a) \(f(x) = x^3 - 12x^2 + 36x\)  (b) \(f(x) = -x^3 + 6x^2 - 9x\)
   (c) \(f(x) = x^3 - 3x^2 + 4\)  (d) \(f(x) = x^3 - 12x - 16\)
   (e) \(f(x) = x^3 - 2x^2 - 4x + 8\)  (f) \(g(x) = -2x^3 + x^2 + 8x - 4\)
   (g) \(g(x) = 1 - x^3\)  (h) \(g(x) = x^3 - 3x^2 + 3x - 1\)
   (i) \(f(x) = 6x^2 - x^3\)  (j) \(f(x) = 2x^3 - 3x^2 + 2x - 3\)

2. The function \(g\) defined by \(g(x) = ax^3 + bx^2 + cx + d\) has the following
   properties. Use this information to draw a neat sketch graph of \(g\) without
   actually solving for \(a, b, c\) and \(d\).
   \(g(0) = 32, \quad g(4) = 0, \quad g(-2) = 0, \quad g'(0) = 0, \quad g'(4) = 0\)
   \(g'(x) > 0\) if \(x < 0\) or \(x > 4\)
   \(g'(x) < 0\) if \(0 < x < 4\)
FINDING THE EQUATION OF A CUBIC FUNCTION

EXAMPLE 16

Determine the equation of
\[ g(x) = ax^3 + bx^2 + cx + d \]
if \( g \) passes through the points \((-2; 0), (-1; 0), (4; 0) \) and \((0; -8)\).
The diagram is not drawn to scale.

Solution

In factorised form the cubic equation \( g(x) = ax^3 + bx^2 + cx + d \) would become
\[ g(x) = a(x - x_1)(x - x_2)(x - x_3) \]
where \( x_1, x_2 \), and \( x_3 \) represent the \( x \)-intercepts (roots) of the cubic function. Also, the constant term \( d \) represents the \( y \)-intercept of the function.

Using this information, it is possible to determine the equation of the given cubic function.

\[ y = a(x - x_1)(x - x_2)(x - x_3) \]
\[ \therefore y = a(x - (2))(x - (-1))(x - 4) \]
\[ \therefore y = a(x + 2)(x + 1)(x - 4) \]

Now substitute \((0; -8)\) into this equation to get the value of \( a \).
\[ \therefore -8 = a(0 + 2)(0 + 1)(0 - 4) \]
\[ \therefore -8 = a(2)(1)(-4) \]
\[ \therefore -8 = -8a \]
\[ \therefore 8a = 8 \]
\[ \therefore a = 1 \]

Substitute \( a = 1 \) into the equation and then expand and simplify to get the equation of the cubic function.
\[ y = 1(x + 2)(x + 1)(x - 4) \]
\[ \therefore y = (x + 2)(x^2 - 3x - 4) \]
\[ \therefore y = x^3 - 3x^2 - 4x + 2x^2 - 6x - 8 \]
\[ \therefore g(x) = x^3 - x^2 - 10x - 8 \]

EXAMPLE 17

Determine the equation of
\[ f(x) = -x^3 + ax^2 + bx + c \]

Solution

Since the coefficient of the term in \( x^3 \) is given \( (a = -1) \), we can write the equation of the cubic in the following form:
\[ y = -l(x-x_1)(x-x_2)(x-x_3) \]
\[ \therefore y = -l(x-(-3))(x-2)(x-2) \]
\[ \therefore y = -l(x+3)(x-2)^2 \]
\[ \therefore y = -(x+3)(x^2 - 4x + 4) \]
\[ \therefore y = -(x^3 - 4x^2 + 4x + 3x^2 - 12x + 12) \]
\[ \therefore y = -(x^3 - x^2 - 8x + 12) \]
\[ \therefore y = -x^3 + x^2 + 8x - 12 \]
\[ \therefore f(x) = -x^3 + x^2 + 8x - 12 \]

**EXAMPLE 18**

(2 ; 9) is a turning point on the graph of \( f(x) = ax^3 + 5x^2 + 4x + b \). Determine the value of \( a \) and \( b \) and hence the equation of the cubic function.

**Solution**

At the turning point (2 ; 9) , we know that \( f'(2) = 0 \)

\[ f(x) = ax^3 + 5x^2 + 4x + b \]
\[ \therefore f'(x) = 3ax^2 + 10x + 4 \]
\[ \therefore f'(2) = 3a(2)^2 + 10(2) + 4 \]
\[ \therefore f'(2) = 12a + 24 \]
\[ \therefore 0 = 12a + 24 \]
\[ \therefore -12a = 24 \]
\[ \therefore a = -2 \]

We can now substitute \( a = -2 \) into the original equation:

\[ y = -2x^3 + 5x^2 + 4x + b \]

In order to get the value of \( b \), substitute the point (2 ; 9) into this equation:

\[ \therefore 9 = -2(2)^3 + 5(2)^2 + 4(2) + b \]
\[ \therefore 9 = -16 + 20 + 8 + b \]
\[ \therefore 9 = 12 + b \]
\[ \therefore b = -3 \]

The equation of the cubic function is therefore \( f(x) = -x^3 + 5x^2 + 4x - 3 \)

**EXERCISE 7**

1. (a) \( f(x) = ax^3 + bx^2 + cx + d \) is the graph of a cubic function passing through the points (6 ; 0) , (0 ; -12) , (1 ; 0) , (-2 ; 0) . Determine the values of \( a, b, c \) and \( d \).

   (b) If the graph of \( f(x) = -x^3 + ax^2 + bx - 12 \) passes through \((-3 ; 0)\) and has a turning point at \((2 ; 0)\) , determine the values of \( a \) and \( b \).

2. (a) Determine \( a \) and \( b \) if \( y = -13 \) is a tangent to the graph of the function \( y = -ax^3 - 4x^2 + 8x + b \) at \( x = -2 \).
(b) The graph of \( f(x) = ax^3 + bx^2 \) has a local turning point at \((2; -4)\). Find the value of \(a\) and \(b\).

(c) The graph of \( y = -x^3 + ax^2 + bx \) has a local turning point at \((2; -32)\). Find the value of \(a\) and \(b\).

(d) The graph of \( y = mx^3 - 3x^2 - 12x + n \) has a local minimum turning point at \((2; -3)\). Find the value of \(m\) and \(n\).

(e) \((1; -10)\) is a local turning point of the graph of the function \( f(x) = -\frac{1}{3} x^3 + bx^2 + c \). Determine the value of \(b\) and \(c\).

(f) The line \( y = -6x - 2 \) is a tangent to the graph of \( f(x) = ax^3 + bx^2 \). Find the value of \(a\) and \(b\) if the point of tangency is \((-1; 4)\).

(g) The line \( 2y + 3x = 6 \) is a tangent to the curve \( g(x) = 2ax^3 - bx^2 \) at \(x = 1\). Determine the value of \(a\) and \(b\).

(h) The gradient of the tangent to \( y = ax^3 + bx^2 \) at the point \((1; 5)\) is 12.
   (1) Determine \(a\) and \(b\).
   (2) Calculate the coordinates of the points on the curve where the tangent to the curve is parallel to the \(x\)-axis.

**THE GRAPH OF THE DERIVATIVE OF A FUNCTION**

Sometimes it is useful to compare the graph of a cubic function with the actual graph of its derivative as well as the graph of the second derivative of the cubic function. Interesting properties can be explored using this strategy.

**EXAMPLE 19**

Consider the graphs of the following functions:

\[
\begin{align*}
f(x) &= x^3 - 6x^2 + 9x \quad \text{(A cubic graph)} \\
f'(x) &= 3x^2 - 12x + 9 \quad \text{(A parabola which is the derivative of the cubic graph)} \\
f''(x) &= 6x - 12 \quad \text{(A line which is the second derivative of the cubic or the derivative of the parabola)}
\end{align*}
\]

The graphs of these functions are drawn below. Discuss the relationships between the three functions.
Solutions

The cubic graph \( y = f(x) \) has turning points at \( x = 1 \) or \( x = 3 \).
At these values of \( x \) it is clear that the derivative of the cubic graph equals 0,
i.e. \( f'(x) = 0 \). The parabola \( y = f'(x) \) cuts the \( x \)-axis at \( x = 1 \) or \( x = 3 \).
At these values of \( x \) the \( y \)-values of the parabola graph are equal to 0
i.e. \( f'(x) = 0 \).
Therefore the cubic graph has turning points where the parabola cuts the \( x \)-axis.

For \( x < 1 \) or \( x > 3 \), the cubic graph increases, i.e. \( f'(x) > 0 \)
The parabola is above the \( x \)-axis for \( x < 1 \) or \( x > 3 \), i.e. \( f'(x) > 0 \)
Therefore the cubic graph increases for the same values of \( x \) for which the parabola
is above the \( x \)-axis.

For \( 1 < x < 3 \), the cubic graph decreases, i.e. \( f'(x) < 0 \)
The parabola is below the \( x \)-axis for \( 1 < x < 3 \), i.e. \( f'(x) < 0 \)
Therefore the cubic graph decreases for the same values of \( x \) for which the parabola
is below the \( x \)-axis.

There is a local maximum turning point at \( x = 1 \) since the cubic graph increases to
the left of 1 (parabola is above the \( x \)-axis) and decreases to the right of 1 (parabola
is below the \( x \)-axis).

There is a local minimum turning point at \( x = 3 \) since the cubic graph decreases to
the left of 1 (parabola is below the \( x \)-axis) and increases to the right of 1 (parabola
is above the \( x \)-axis).

At \( x = 2 \) the cubic graph has a point of inflection. This is where the line \( y = f''(x) \)
cuts the \( x \)-axis or where the parabola has its turning point, i.e. where \( f''(x) = 0 \).

EXAMPLE 20

In the diagram, the graph of \( y = f'(x) \) is
given where \( f(x) = ax^3 + bx^2 + cx \) represents a
cubic function.
(a) Determine the equation of \( f'(x) \).
(b) Determine the equation of \( f(x) \).
(c) By referring to the diagram, determine
the values of \( x \) for which the graph of
\( f(x) \) has its stationary points.
(d) By referring to the diagram, determine the values of \( x \) for which the graph
of \( f(x) \) is increasing and decreasing.

Solutions

(a) The graph represented is a parabola with \( x \)-intercepts of 1 and 2.
One of the forms of the equation of a parabola is given by
\( y = a(x - x_1)(x - x_2) \) where \( x_1 \) and \( x_2 \) are the roots of the parabola.
\( \therefore y = a(x - 1)(x - 2) \)
Now substitute the point \((0;12)\) into the equation to get the value of \( a \):
\[ \therefore 12 = a(0 - 1)(0 - 2) \]
\[ \therefore 12 = a(-1)(-2) \]
\[ \therefore 12 = 2a \]
\[ \therefore -2a = -12 \]
\[ \therefore a = 6 \]
Substitute \( a = 6 \) into the equation to get the equation of \( y = f'(x) \):
\[ y = 6(x - 1)(x - 2) \]
\[ \therefore y = 6(x^2 - 3x + 2) \]
\[ \therefore y = 6x^2 - 18x + 12 \]
\[ \therefore f'(x) = 6x^2 - 18x + 12 \]

(b) We know that \( f(x) = ax^3 + bx^2 + cx \).
\[ \therefore f'(x) = 3ax^2 + 2bx + c \]
But we have already determined the equation of \( f'(x) \):
\[ f'(x) = 6x^2 - 18x + 12 \]
By equating the coefficients of the two expressions for \( f'(x) \), we get:
\[ 3a = 6 \quad 2b = -18 \quad c = 12 \]
\[ \therefore a = 2 \quad \therefore b = -9 \]
\[ f(x) = 2x^3 - 9x^2 + 12x \]

(c) The graph of \( f(x) \) has stationary points where \( f'(x) = 0 \). The graph of \( f'(x) \) cuts the \( x \)-axis at \( x = 1 \) and \( x = 2 \). At these points, \( f'(x) = 0 \).
Therefore the graph of \( f(x) \) has stationary points at \( x = 1 \) and \( x = 2 \).

(d) The graph of \( f(x) \) increases for all values of \( x \) for which \( f'(x) > 0 \).
The parabola lies above the \( x \)-axis for \( x < 1 \) or \( x > 2 \). Therefore, the graph of \( f(x) \) increases for \( x < 1 \) or \( x > 2 \).
The graph of \( f(x) \) decreases for all values of \( x \) for which \( f'(x) < 0 \).
The parabola lies below the \( x \)-axis for \( 1 < x < 2 \). Therefore, the graph of \( f(x) \) decreases for \( 1 < x < 2 \).

**EXERCISE 8**

1. In the diagram alongside, the graph of \( y = f'(x) \) is represented where
\[ f(x) = ax^3 + bx^2 + cx \] represents a cubic function.
(a) Determine the equation of \( f'(x) \) and \( f(x) \) where
\[ f(x) = ax^3 + bx^2 + cx \]
(b) For which values of \( x \) will the graph of \( f(x) \)
(1) increase?
(2) decrease?
(c) At which values of \( x \) will the graph of \( f \) have its turning points?
2. In the diagram alongside, the graph of \( y = f'(x) \) is represented, which is the graph of the derivative of the cubic function \( f(x) \).

(a) What is the gradient of the tangent to the graph of \( f(x) \) at \( x = 0 \)?

(b) For which value of \( x \) will there be a tangent to the curve of \( f \) which will be parallel to the tangent in (a)?

(c) For which values of \( x \) will \( f(x) \) be decreasing?

(d) Write down the \( x \)-coordinates of the turning points of \( f \) and state whether they are local maximum or minimum turning points.

(e) If it is further given that the \( x \)-intercepts of the graph of \( f \) are \(-2, 2 \) and \( 7 \), use the information at your disposal and draw the graph of \( f \). It is unnecessary to determine the \( y \)-coordinates of the turning points.

**FINDING THE EQUATION OF A TANGENT**

- The gradient of the tangent \((t)\) to the graph of a function \( f \) at the point \( T(x_T; y_T) \) is given by \( m_t = f'(x_T) \).
- We will use the equation \( y - y_T = m_t(x - x_T) \) to determine the equation of the tangent at \( T \).

**EXAMPLE 21**

The graph of \( g(x) = -3x^3 + 2x^2 - 3x + 5 \) is represented in the diagram below. The equation of the tangent to the graph of \( g \) is drawn and the point of tangency is at the point \( T \) where \( x = -1 \). The graphs intersect at \( T \) and \( B \).
(a) Determine the equation of the tangent to \( g \) at \( x = -1 \).

(b) Find the coordinates of the point where the tangent cuts the curve of \( g \) again.

**Solutions**

(a) In order to get the equation of the tangent to \( g \) at \( x = -1 \), we need the following:

\[ x_T, \ y_T, \ \text{and} \ m_t \]

We are given that \( x_T = -1 \).

To get \( y_T \), substitute \( x = -1 \) into the original equation:

\[ y_T = g(-1) = -3(-1)^3 + 2(-1)^2 - 3(-1) + 5 \]
\[ \therefore y_T = 3 + 2 + 3 + 5 \]
\[ \therefore y_T = 13 \]

To get \( m_t \), determine \( g'(-1) \):

\[ g(x) = -3x^3 + 2x^2 - 3x + 5 \]
\[ \therefore g'(x) = -9x^2 + 4x - 3 \]
\[ \therefore g'(-1) = -9(-1)^2 + 4(-1) - 3 \]
\[ \therefore g'(-1) = -9 - 4 - 3 \]
\[ \therefore g'(-1) = -16 \]
\[ \therefore m_t = -16 \]

(b) We determine where the graph of the tangent and the graph of the cubic function intersect.

\[ -16x - 3 = -3x^3 + 2x^2 - 3x + 5 \]
\[ \therefore 3x^3 - 2x^2 - 13x - 8 = 0 \]
\[ \therefore (x + 1)(3x^2 - 5x - 8) = 0 \]
\[ \therefore (x + 1)(3x - 8)(x + 1) = 0 \]
\[ \therefore x = -1 \ \text{or} \ x = \frac{8}{3} \]

The \( x \)-coordinate of B is \( x = \frac{8}{3} \). To get the corresponding \( y \)-coordinate, substitute \( x = \frac{8}{3} \) into either the equation of the cubic or the tangent.

Obviously, the equation of the tangent is the easier equation to use.

\[ y = -16x - 3 \]
\[ \therefore y = -16(\frac{8}{3}) - 3 \]
\[ \therefore y = -\frac{128}{3} - 3 \]
\[ \therefore y = -\frac{137}{3} = -\frac{45\frac{2}{3}}{3} \]

The coordinates of B are \( \left( \frac{2}{3}; -\frac{45\frac{2}{3}}{3} \right) \).
EXAMPLE 22

Determine the equation of the tangent to \( f(x) = x^2 - 2x + 1 \) if the gradient of the tangent is negative and the \( y \)-coordinate of the point of tangency is 4.

Solution

In this example, we are given \( y_T \) and need to find \( x_T \) and \( m_t \).

In order to get \( x_T \), substitute \( y = 4 \) into the original equation:
\[
y_T = 4 \\
\therefore 4 = x^2 - 2x + 1 \\
\therefore 0 = x^2 - 2x - 3 \\
\therefore 0 = (x-3)(x+1) \\
\therefore x = 3 \quad \text{or} \quad x = -1
\]

In order to get \( m_t \), determine \( f'(x) \) and then substitute the two \( x \)-values into this expression to determine the gradient of the tangent line.
\[
f(x) = x^2 - 2x + 1 \\
\therefore f'(x) = 2x - 2
\]

For \( x = 3 \), \( f'(3) = 2(3) - 2 = 4 \) \quad \text{For} \quad x = -1, \quad f'(-1) = 2(-1) - 2 = -4

But since the gradient is given as negative, we know that for the required equation of the tangent:
\[
x_T = -1, \quad y_T = 4 \quad \text{and} \quad m_t = -4
\]

\[
y - y_T = m_t(x - x_T) \\
\therefore y - 4 = -4(x - (-1)) \\
\therefore y - 4 = -4(x + 1) \\
\therefore y - 4 = -4x - 4 \\
\therefore y = -4x
\]

EXAMPLE 23

Determine the value(s) of \( p \) if the line \( y = 3x + p \) is a tangent to the graph of \( f(x) = 2x^3 - 3x - 1 \).

Solution

In this example, we are given \( m_t = 3 \) since the gradient of the tangent line \( y = 3x + p \) is equal to the coefficient of \( x \), which is 3.

\[
f(x) = 2x^3 - 3x - 1 \\
\therefore f'(x) = 6x^2 - 3 \\
\therefore 3 = 6x^2 - 3 \quad \text{(the gradient of the tangent is 3)} \\
\therefore 0 = 6x^2 - 6 \\
\therefore 0 = x^2 - 1 \\
\therefore 1 = x^2 \\
\therefore x = \pm 1
\]
We can now determine the corresponding $y$-values:

For $x = 1$

$$f(1) = 2(1)^3 - 3(1) - 1$$

$$f(-1) = 2(-1)^3 - 3(-1) - 1$$

$$\therefore f(1) = -2 \quad \therefore f(-1) = 0$$

We have two tangents:

For $x_T = 1, \quad y_T = -2 \quad \text{and} \quad m_T = 3$

$$y - y_T = m_T(x - x_T)$$

$$y - (-2) = 3(x - 1)$$

$$\therefore y = 3x - 3$$

$$\therefore y = 3x + 5$$

EXERCISE 9

1. (a) Determine the equation of the tangent to $f(x) = x^2 - 6x + 5$ at $x = 2$

(b) Determine the equation of the tangent to the curve $y = 2x^3 - 21x^2 + 59x - 20$ at $x = 5$.

2. (a) Determine the equation of the tangent to the curve $y = 3x^2 - 2x + 2$ at $x = -4$.

(b) Hence determine where this tangent cuts the $x$-axis.

3. Determine the equation of the tangent to the curve $y = x^3 - x^2 - 35x - 50$ at the point where $x = -3$. Find the coordinates of the point where the tangent meets the curve again.

4. Determine the equation of the tangent to:

(a) $y = \frac{9}{x}$ at $x = 1$

(b) $y = 2\sqrt{x}$ at $x = 9$

(c) $y = \frac{4x^2 - 1}{2x - 1}$ at $x = 2$

(d) $f(x) = \frac{x^3 - 2x^2 + 3}{x}$ at $x = -1$

5. (a) Determine the equation of the tangent to the curve $g(x) = x^2 + 4x - 5$ if the gradient of the tangent is negative and the $y$-coordinate of the point of tangency is $-8$.

(b) Determine the equation of the tangents to the curve $f(x) = x^2 - 3x - 4$ at the points where $f(x) = 0$.

6. (a) Find the point on the graph of $y = x^2$ for which the gradient is 6.

(b) Find the point on the graph of $y = 4\sqrt{x}$ for which the slope is 1.

(c) Find the points on the graph of $xy = 4$ for which the gradient is $-4$.

(d) Find the point on the graph of $y = x^2 - 5x - 1$ for which the gradient is $-2$.

7. (a) Determine the equation of the tangent to the curve $f(x) = -x^2 + 3x$ which is parallel to the line $y = x + 2$.

(b) Find the value of $p$ if $y = p - 9x$ is a tangent to $y = -x^3 + 3x - 2$.

(c) Find the value of $t$ if the line $y + 2x = t$ is a tangent to the curve $f(x) = 5 + 4x - x^2$.

(d) The graph of $f(x) = x^2 + 6$ has a tangent at $x = 3$. The tangent passes through the point $(2; 1)$. Determine the equation of the tangent.

(e) The tangent to $f(x) = 4 - x^2$ at $(a; f(a))$ passes through $(4; -3)$. Determine the equations of the tangents.
(f) At what point on the graph of \( f(x) = 3x^2 \) does the tangent to \( f(x) \) form an angle of 45° with the positive x-axis?

8. The graph of \( f(x) = ax^2 + bx \) passes through the point \( P(3; 6) \) and cuts the x-axis at \( (4; 0) \).
   Determine the equation of the tangent to \( f \) at \( P \).

PROBLEMS INVOLVING MAXIMUM AND MINIMUM VALUES

We now focus on some real world applications of calculus. For example, we can find maximum or minimum area (or volume) as well as minimize variables like cost, perimeter and so forth. Before discussing the examples on maximum and minimum values, it is advisable for you to complete the investigations which explore the basic principles (these may be photocopied from the Teacher’s Guide page 211).

**Rule for determining maximum or minimum values**

In order to maximize or minimize an object \( ax \), determine \( a'(x) \), equate \( a'(x) \) to zero and solve for \( x \). The value(s) of \( x \) obtained can then be investigated to establish whether they will yield maximum or minimum values of that given object.

**TWO DIMENSIONAL PROBLEMS**

**EXAMPLE 24**

A rectangular area is to be enclosed on three sides by a fence and an existing hedge which forms the fourth side.

(a) Find the maximum area of the rectangular area and its dimensions if 100 metres of fencing is available.
(b) The cost of fencing is R20 per metre. Suppose that the rectangular area must be 200 \( m^2 \). What are the dimensions such that the cost will be a minimum?
Solutions

(a) We need to first find an expression for the rectangular area.

Area = length \times \text{breadth}

\therefore \text{Area} = y \times x

Since 100 metres of fencing are available and the hedge serves as one of the sides, we know that

\begin{align*}
  x + y + x &= 100 \\
  \therefore y + 2x &= 100 \\
  \therefore y &= 100 - 2x
\end{align*}

We now need to get the area expression in terms of \(x\) before differentiating:

\begin{align*}
  \text{Area} &= y \times x \\
  \therefore A(x) &= (100 - 2x)x \\
  A(x) &= 100x - 2x^2
\end{align*}

We now need to apply the golden rule for maxima and minima:

\begin{align*}
  A(x) &= 100x - 2x^2 \\
  \therefore A'(x) &= 100 - 4x \\
  \therefore 0 &= 100 - 4x \quad (A'(x) = 0 \text{ for a max or min}) \\
  \therefore 4x &= 100 \\
  \therefore x &= 25
\end{align*}

Since \(A(x) = 100x - 2x^2\) represents an “unhappy” parabola with a maximum value, it is therefore clear that the maximum occurs at \(x = 25\).

Therefore the maximum area can be calculated by substituting this value of \(x\) into the original equation:

\begin{align*}
  A(x) &= 100x - 2x^2 \\
  \therefore A(25) &= 100(25) - 2(25)^2 = 1250m^2
\end{align*}

The dimensions of the rectangular area which produces a maximum area are: Breadth = 25m Length = \(y = 100 - 2(25) = 50m\)

(b) Cost = 20\(x + 20x + 20y\)

Cost = 40\(x + 20y\)

We are given that: Area = 200

\therefore xy = 200

\therefore y = \frac{200}{x}

\begin{align*}
  C(x) &= 40x + 20\left( \frac{200}{x} \right) \\
  \therefore C(x) &= 40x + \frac{4000}{x}
\end{align*}

\therefore C(x) = 40x + 4000x^{-1}

In order to minimize cost, proceed as follows:
\[ C(x) = 40x + 4000x^{-1} \]
\[ \therefore C'(x) = 40 - 4000x^{-2} \]
\[ \therefore C'(x) = 40 - \frac{4000}{x^2} \]
\[ \therefore 0 = 40 - \frac{4000}{x^2} \]
\[ \therefore 0 = 40x^2 - 4000 \]
\[ \therefore 0 = x^2 - 100 \]
\[ \therefore x^2 = 100 \]
\[ \therefore x = \pm 10 \]
Clearly, the value of \( x \) cannot be negative in this example.
\[ \therefore x = 10m \]
We can now calculate the value of \( y \):
\[ y = \frac{200}{x} \]
\[ \therefore y = \frac{200}{10} = 20m \]
Therefore, the dimensions which will minimize the cost are:
Length = 20m  Breadth = 10m

**EXAMPLE 25**

The area of the figure given is 20 \( m^2 \).

(a) Find an expression for the total area in terms of \( \pi, r \), and \( l \).

(b) Show that \( l = \frac{40 - \pi r^2}{4r} \)

(c) Show that the perimeter \( P \) of the figure is given by:
\[ P = \frac{20}{r} + 2r + \frac{\pi}{2}r \]

(d) Calculate the width of the figure when \( P \) is a minimum. Round off your answer to one decimal place.

**Solutions**

(a) \( A = \) area of rectangle + area of semi-circle
\[ \therefore A = l \times 2r + \frac{1}{2} \pi r^2 \]
\[ \therefore A = 2lr + \frac{\pi}{2}r^2 \]
(b) Since the area of the figure is given as $20 m^2$, we can substitute this value into the equation in (a) and then proceed to make $l$ the subject of the formula.

\[ A = 2lr + \frac{\pi}{2} r^2 \]

\[ \therefore 20 = 2lr + \frac{\pi}{2} r^2 \]

\[ \therefore 40 = 4lr + \pi r^2 \]

\[ \therefore 40 - \pi r^2 = 4lr \]

\[ \therefore \frac{40 - \pi r^2}{4r} = l \]

\[ \therefore l = \frac{40 - \pi r^2}{4r} \]

(c) \[ P = l + l + 2r + \frac{1}{2} \times 2\pi r \]

\[ \therefore P = 2l + 2r + \pi r \]

\[ \therefore P = 2 \left( \frac{40 - \pi r^2}{4r} \right) + 2r + \pi r \]

\[ \therefore P = \frac{40 - \pi r^2}{2r} + 2r + \pi r \]

\[ \therefore P = \frac{40}{2r} - \frac{\pi r^2}{2r} + 2r + \pi r \]

\[ \therefore P = \frac{20}{r} - \frac{\pi r^2}{r} + \pi r + 2r \]

\[ \therefore P = \frac{20}{r} + \frac{2\pi r}{2} + \frac{\pi r}{2} + 2r \]

\[ \therefore P = \frac{20}{r} + \frac{2\pi r}{2} + \frac{\pi r}{2} + 2r \]

\[ \therefore P = \frac{20}{r} + 2r + \frac{\pi r}{2} \]

\[ \therefore P = \frac{20}{r} + 2r + \frac{\pi r}{2} \]

(d) \[ P = \frac{20}{r} + 2r + \frac{\pi}{2} r \]

\[ \therefore P = 20r^{-1} + 2r + \frac{\pi}{2} r \]

\[ \therefore P'(r) = -20r^{-2} + 2 + \frac{\pi}{2} \]

\[ \therefore P'(r) = \frac{-20}{r^2} + 2 + \frac{\pi}{2} \]

\[ \therefore 0 = \frac{-20}{r^2} + 2 + \frac{\pi}{2} \]

\[ \therefore 0 = -40 + 4r^2 + \pi r^2 \]

\[ \therefore 40 = r^2 (4 + \pi) \]

\[ \therefore \frac{40}{4 + \pi} = r^2 \]

\[ \therefore r = \sqrt{\frac{40}{4 + \pi}} \]

\[ \therefore r = 2.4 \]

Therefore, the width is $4.8$ metres.
EXERCISE 10

1. A farmer has 100 metres of wire fencing from which to build a rectangular chicken run. He intends using two existing adjacent walls for two sides of the rectangular enclosure. Determine the dimensions which give a maximum enclosed area.

2. A toll-road company is planning to build a picnic site along a major highway. It is to be rectangular in shape with an area of $5000 \text{m}^2$ and is to be fenced off on the three sides adjacent to the highway. Let the length be $x$ and the width be $y$.
   (a) Find the total length, $L$, of fencing required in terms of $x$ and $y$.
   (b) Show that the total length of fencing required is also given by the formula:
   $$L = x + \frac{10000}{x}$$
   (c) Find the least amount of fencing that will be needed to complete the job.

3. For the latest issue of a magazine, open spaces of 10mm must be left at the bottom and top of the page and 15 mm spaces at the sides. The printed part must cover an area of $5400 \text{mm}^2$. Assume that $HG = x \text{mm}$.
   (a) Write down the length of $BC$ and $AB$ in terms of $x$.
   (b) Hence determine the length of $BC$ and $AB$ such that the area of page $ABCD$ will be a minimum.

4. The owner of a plot of land wishes to fence a rectangular area of $432 \text{m}^2$ on his plot. One side of this rectangular area borders on his neighbour’s property. The neighbour is prepared to pay half of the cost of the fence on the boundary line.
   (a) If one side of the rectangular area is $x$ metres and the cost of erecting the fence is R15 per metre, express the total cost ($T$) to the owner of the plot in terms of $x$.
   (b) Hence calculate the dimensions of the rectangular area if the owner of the plot wishes to pay the minimum for erecting the fence.
5. A new design for a rectangular table place mat has been implemented by a company manufacturing dining room items. On the two width sides of the rectangular place mat, two white semi-circles are pasted as shown in the diagram below. The remaining area has been shaded with a multi-coloured pattern. The length of the place mat remains fixed at 100mm, while the width of the mat can vary in length.

(a) Show that the shaded area is given by $A(x) = 100x - \frac{\pi}{4}x^2$.

(b) Determine the maximum shaded area possible.

6. The top view of a dam containing fish is to be of the shape as shown in the diagram below. The shape consists of four semi-circles, two with a radius of $R$ metres and two with a radius of $r$ metres. The values of $R$ and $r$ may vary, but the sum $R + r$ remains fixed at 200 metres. Determine the values of $R$ and $r$ if the surface area of the top view is to be a minimum.

7. A metal frame consisting of four congruent rectangles and a semi-circle must be manufactured. The total length of the material that is to be used for the whole frame is 36 metres.

(a) If the length of a rectangle is $x$ metres and the width is $h$ metres, show that the total length of the material is given by: $L = (6h + 6x + \pi x)$ metres.

(b) Hence show that the area of the frame is given by: $A = (24x - 4x^2 - \frac{1}{4}\pi x^2)$ metres$^2$.

(c) Hence determine the length of the base of the frame if the frame is to have a maximum area.
THREE DIMENSIONAL PROBLEMS

EXAMPLE 26

A cylinder, closed at both ends, is to have a volume of \((2000\pi)\text{cm}^3\). What should its dimensions be if its surface area is to be as small as possible?

Solution

Surface Area \((A) = 2\pi r^2 + 2\pi rh\)

\[\therefore A = 2\pi r^2 + 2\pi rh\]

Now the formula for volume is given by \(V = \pi r^2 h\) and since the actual volume is given as \((2000\pi)\text{cm}^3\), we can substitute this value into the formula and then make \(h\) the subject of the formula.

\[2000\pi = \pi r^2 h\]

\[\therefore 2000 = r^2 h\]

\[\therefore \frac{2000}{r^2} = h\]

\[\therefore h = \frac{2000}{r^2}\]

Now substitute this expression for \(h\) into the formula for surface area:

\[A(r) = 2\pi r^2 + 2\pi r \left(\frac{2000}{r^2}\right)\]

\[\therefore A(r) = 2\pi r^2 + \frac{4000\pi}{r}\]

\[\therefore A(r) = 2\pi r^2 + 4000\pi r^{-1}\]

\[\therefore A'(r) = 4\pi r - 4000\pi r^{-2}\]

\[\therefore A'(r) = 4\pi r - \frac{4000\pi}{r^2}\]

For a minimum value of \(r\):

\[0 = 4\pi r - \frac{4000\pi}{r^2}\]

\[\therefore 0 = 4\pi r^3 - 4000\pi\]

\[\therefore 0 = r^3 - 1000\]

\[\therefore 1000 = r^3\]

\[\therefore r = 10\text{cm}\]

We can now calculate the value of \(h\) corresponding to this value of \(r\):

\[h = \frac{2000}{r^2}\]

\[\therefore h = \frac{2000}{(10)^2} = \frac{2000}{100} = 20\text{cm}\]

The minimum surface area will be:

\[A = 2\pi(10)^2 + 2\pi(10)(20)\]

\[\therefore A = 200\pi + 400\pi\]

\[\therefore A = (600\pi)\text{cm}^2\]
EXERCISE 11

1. A piece of rectangular sheet metal with dimensions 80 mm by 50 mm is used to create a rectangular jewelry box. Square corners of \( x \) mm are cut out of the sheet metal and the edges are then folded up so as to form a box without a lid of depth \( x \) mm.

(a) Show that the volume of the jewelry box is given by the formula:
\[
V(x) = 4x^3 - 260x^2 + 4000x.
\]

(b) Determine the value of \( x \) so that the volume of the box is a maximum.

2. Bricks are to be painted with a special paint for use under water. The bricks are rectangular in shape and the length of each brick is three times its breadth. The volume of cement in each brick is 972 cm\(^3\). If \( h \) is the height of a brick and \( x \) is its breadth:

(a) express \( h \) in terms of \( x \).

(b) express the total surface area of each brick in terms of \( x \).

(c) calculate the dimensions of a brick which will minimize the amount of paint required.

3. The sides of the base of a rectangular cardboard shoe box are 3\( x \) and 2\( x \) cm respectively. The height is \( y \) cm. The box is open on the top (without a lid at this stage).

(a) If the total surface area of the box is 200 cm\(^2\), prove that
\[
y = \frac{20}{x} - \frac{3x}{5}.
\]

(b) Express the volume of the box in terms of \( x \).

(c) Determine the dimensions of the rectangular cardboard box if its volume is to be a maximum.

4. A rectangular water tank is to be manufactured. It must contain 32 m\(^3\) of water. It has a square base (each side equal to \( x \) metres). The top of the water tank is open and its height is \( h \) metres.

(a) Determine the area (\( A \)) of the material that will be used in terms of \( x \) and \( h \).

(b) Prove that \( A = x^2 + \frac{128}{x} \)

(c) Find, in terms of \( x \), an expression for \( C \), the total cost of the material, if the material is bought at a price of R10 per m\(^2\).

(d) Determine the values of \( x \) and \( h \) for which \( C \) will be a minimum.
5. A solid cylinder is cast from 10 litres of molten metal. This cylinder is then covered by a layer of rust-proof paint. Calculate the radius and the height of the cylinder (in cm) such that the minimum quantity of paint is to be used.

6. The total exterior surface area of an empty cylinder, which is closed on one end, is $462 \, \text{cm}^2$. 
   (a) Find the height ($h$) of the cylinder in terms of $\pi$ and radius ($r$).
   (b) Hence find the value of $r$ for which the volume of the cylinder is a maximum.

7. The diagram represents a right circular cone of height, $h$ cm, and a base radius of $r$ cm. If the sum of the height and base radius is 12 cm,
   (a) express $r$ in terms of $h$.
   (b) express the volume of the cone in terms of $h$.
   (c) determine the maximum volume of the cone.

**EXAMPLE 27**

In the diagram below, $f(x) = x^3 - 3x - 2$ and $g(x) = 2x - 2$. Line PQ is parallel to the $y$-axis and varies in length between B and C. Show that the value of $x$ for which the length of PQ is a maximum is given by $x = \frac{\sqrt{5}}{3}$.

**Solution**

\[ PQ = y_p - y_Q \]
\[ \therefore PQ = (2x - 2) - (x^3 - 3x - 2) \]
\[ \therefore PQ = 2x - 2 - x^3 + 3x + 2 \]
\[ \therefore PQ = -x^3 + 5x \]
\[ \therefore \frac{d}{dx} (PQ) = -3x^2 + 5 \]
At a maximum, the derivative equals zero.
\[ \therefore 0 = -3x^2 + 5 \]
\[ \therefore 3x^2 = 5 \]
\[ \therefore x^2 = \frac{5}{3} \]
\[ \therefore x = \sqrt{\frac{5}{3}} \quad \text{(since } x > 0) \]
EXAMPLE 28

A rectangle PQRS is drawn with P and Q on the x-axis and R and S on the parabola

\[ y = -2x^2 + 24. \]

If Q is the point \( (x; 0) \), determine:

(a) the coordinates of P and R in terms of \( x \).
(b) PQ and QR in terms of \( x \).
(c) the area of PQRS in terms of \( x \).
(d) the area of the largest rectangle that can be drawn.

Solutions

(a) By symmetry, the point P is the reflection of Q about the y-axis.

Therefore the point P is \((-x; 0)\).

The coordinates of point R are \((x; -2x^2 + 24)\).

(b) PQ = 2x

QR = \(-2x^2 + 24\)

(c) Area of PQRS = 2x(-2x^2 + 24)

\[ \therefore A(x) = -4x^3 + 48x \]

(d) \[ A(x) = -4x^3 + 48x \]

\[ \therefore A'(x) = -12x^2 + 48 \]

\[ \therefore 0 = -12x^2 + 48 \]

\[ \therefore 12x^2 = 48 \]

\[ \therefore x^2 = 4 \]

\[ \therefore x = 2 \quad (x > 0) \]

\[ \therefore \text{Maximum area of PQRS is:} \]

\[ A(2) = -4(2)^3 + 48(2) = 64 \text{ units}^2 \]

EXERCISE 12

1. In the diagram, the graphs of \( f(x) = 6x - 2x^2 \) and \( g(x) = 2x \) are shown. Line segment RQ is parallel to the y-axis. RQ varies in length between the two points of intersection of the graphs of \( f \) and \( g \).

Determine the maximum length of RQ.
2. The line with equation \( x + y = 4 \) and the graph of the function \( xy = 1 \) for \( x > 0 \) are represented in the diagram alongside. Line segment PQ varies in length between the two points of intersection of the graphs. Determine the maximum length of PQ.

3. In the diagram below, the graph of \( f(x) = 18 - 2x^2 \) is drawn. ABCD is a rectangle drawn within the parabola with A(x; y) and B on the parabola. Point C and D lie on the x-axis. Determine:
   (a) the coordinates of A and B in terms of \( x \).
   (b) the area of ABCD in terms of \( x \).
   (c) the area of the largest possible rectangle.

4. PQRS is a rectangle with sides parallel to the axes. The points P, Q, R and S are points on the parabolas
   \[ y = -\frac{1}{3}x^2 + 4 \] and \( y = \frac{1}{6}x^2 - 2 \).
   (a) Express PQ and QR in terms of \( x \).
   (b) Calculate the maximum area of PQRS.

5. In the diagram below, the graph of the function \( f(x) = 8 - x^3 \) for \( x \geq 0 \) and \( y \geq 0 \) is represented.
   The graph of \( f \) intersects the x-axis at B and the y-axis at A. P(x; 0) is any point on the interval OB. D lies on \( f \) such that CD \( \perp \) AO and DP \( \perp \) OB.
   (a) Determine the length of OA and OB.
   (b) Determine the length of OC in terms of \( x \).
   (c) Prove that the area of quadrilateral OBDA is equal to:
   \[ A(x) = -x^3 + 4x + 8 \] .
   (d) For which value of \( x \) is the area of the quadrilateral is a maximum?
FURTHER APPLICATIONS (RATES OF CHANGE AND MOTION)

By now, it will be clear to you that the rate of change of one variable with respect to another can be represented as a derivative. A rate of change can be positive or negative. For example, if the volume of air in a balloon is increasing and the radius of the spherical balloon is expanding, then the rate of change of volume with respect to the radius of the spherical balloon is positive. However, if air escapes from the balloon, then the rate of change is negative.

EXAMPLE 29

The volume of water in a horse trough is governed by the equation

\[ V(t) = 20t - t^2 \quad \text{for} \quad t \in [0; 20] \]

where

V is the volume of water in m\(^3\) and

\( t \) is time in minutes.

The graph of this equation is represented on the right.

(a) Determine the volume at \( t = 2 \).
(b) Determine the rate of change of volume at \( t = 2 \).
   Explain what is happening by referring to the graph.
(c) Determine the time at which the volume is a maximum.
(d) Determine the rate of change of volume at \( t = 16 \).
   Explain what is happening by referring to the graph.

Solutions

(a) \( V(t) = 20t - t^2 \)
   \[ \therefore V(2) = 20(2) - (2)^2 \]
   \[ \therefore V(2) = 40 - 4 \]
   \[ \therefore V(2) = 36 \text{ m}^3 \]

(b) \( V(t) = 20t - t^2 \)
   \( V'(t) = 20 - 2t \)
   \[ \therefore V'(2) = 20 - 2(2) = 16 \text{ m}^3 / \text{min} \]
   The volume of water in the trough is increasing (the derivative is positive).

(c) \( V'(t) = 20 - 2t \)
   At a maximum, \( V'(t) = 0 \)
   \[ \therefore 0 = 20 - 2t \]
   \[ \therefore 2t = 20 \]
   \[ \therefore t = 10 \text{ mins} \]
Notice that the tangent at \( t = 10 \) is horizontal, which occurs at the maximum turning point on the graph. Here the derivative is zero. The volume is a maximum at \( t = 10 \).

(d) \( V'(t) = 20 - 2t \)
\[ \therefore V'(16) = 20 - 2(16) \]
\[ \therefore V'(16) = -12 \text{ m}^3 / \text{min} \]
The volume of water in the trough is decreasing (the derivative is negative).

**EXERCISE 13**

1. In an experiment, the number of germs in a test tube at any time \( t \), in seconds, is given by the equation \( g(t) = 3t^2 + 2 \). Determine:
   (a) the number of germs in the test tube after 4 seconds.
   (b) the rate of change in the number of germs after 4 seconds.
   (c) the rate of change in the number of germs after 6 seconds.

2. The volume of air in a spherical balloon changes with respect to its radius. Determine the rate of change when the radius is 50 mm in length.

3. The volume of water in a tank is governed by the equation \( V(t) = 10t - t^2 \) for the interval \( t \in [0; 10] \) where \( V \) represents volume in \( \text{m}^3 \) and \( t \) represents the time in minutes. Determine:
   (a) the volume after 4 minutes.
   (b) the time taken for the volume to reach 9 \( \text{m}^3 \).
   (c) an expression for the rate of change of volume.
   (d) the rate of change of volume at \( t = 4, \ t = 5 \) and \( t = 6 \). Explain your findings.
   (e) the time taken to reach a maximum volume.
   (f) the maximum volume.
   (g) the times during which the volume is decreasing.
   (h) the times at which there is no water in the tank.
   (i) the time at which the rate of increase is 6 \( \text{m}^3 / \text{min} \).
   (j) the time at which the rate of decrease is 8 \( \text{m}^3 / \text{min} \).

**Some theory on the calculus of motion**

\( s(t) \) represents an equation of motion (height, distance, displacement) at time \( t \).
\( s'(t) \) represents velocity or speed at time \( t \).
\( s''(t) \) represents acceleration at time \( t \).

Consider the following situation of an object moving from its position at O to A and then back to position C.

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**Diagram:**

```
B   O   C   A
```
If the object moves from O to A, we say that its displacement from O is (+6) units. If the object now moves in the opposite direction to position C, we say that its displacement from A to C is (-4) units. The object’s displacement from position O after moving from O to A and then to C is (6) + (-4) = 2 units. The total distance moved from O to A to C is 10 units. Distance is always positive because it does’t take direction into consideration. Displacement does take direction into consideration and can therefore be positive or negative. 

Velocity is speed in a given direction. If the object moves in one direction, say from left to right (from B to A), we can define its velocity to be positive. If the object moves in the opposite direction (from A to B), the velocity would then be negative. The speed of the object is independent of the direction of movement. 

In the next exercise, you will be required to use these concepts.

**EXERCISE 14**

1. An object moves from its starting position at point O and its motion is described by the equation \( s(t) = 20t - 2t^2 \), where \( s \) represents displacement in metres and \( t \), the time in seconds. Determine: 
   (a) the displacement of the object after 1 second.  
   (b) the time taken to reach maximum displacement for the given interval.  
   (c) the maximum displacement.  
   (d) the times taken to reach a displacement of 32m from O. 
   (e) the distance moved after \( t = 2 \) and \( t = 8 \) seconds.  
   (f) the time taken to reach O again.  
   (g) the times during which the displacement is increasing.  
   (h) the times during which the displacement is decreasing.  
   (i) the initial velocity of the object \( (t = 0) \).  
   (j) the velocity and speed at \( t = 1 \). What can you conclude?  
   (k) the time taken to reach a velocity of 4 metres per second.  
   (l) the velocity and speed at \( t = 5 \). What can you conclude?  
   (m) the velocity and speed at \( t = 6 \). What can you conclude?  
   (n) the time taken to reach a velocity of 16 metres per second in a direction opposite to its starting direction.  
   (o) the acceleration of the object (see theory on previous page). 

2. A ball is thrown into the air and its height, \( h \), above the ground, after \( t \) seconds is \( h = -5t^2 + 25t + 4 \) metres. Determine: 
   (a) the height of the ball above the ground after 2 seconds.  
   (b) the velocity of the ball at that moment.  
   (c) the maximum height of the ball above the ground.  
   (d) the time taken to reach a downwards velocity of 5 m/s.
REVISION EXERCISE

1. Given: \( f(x) = 2 - \frac{1}{2}x^2 \)
   (a) Determine the average gradient of \( f \) between \( x = -2 \) and \( x = 4 \).
   (b) Determine the gradient of \( f \) at \( x = -2 \) by using first principles.
   (c) Determine the equation of the tangent to \( f \) at \( x = -2 \).

2. Determine:
   (a) \( \frac{dy}{dx} \) if \( y = 2\sqrt{x} + \frac{8}{x} \)
   (b) \( \frac{dy}{dx} \) if \( y = \frac{6}{3\sqrt[6]{x}} - \frac{1}{6x^6} \)
   (c) \( D_x \left[ \frac{x^2 - 2x + 3}{x^{-1}\sqrt{x}} \right] \)
   (d) \( D_x \left[ \left( \frac{3}{x} + 4\sqrt{x} \right)^2 \right] \)
   (e) \( \frac{dx}{dy} \) if \( y^2 = \frac{x^2 - 2x}{4x - 8} \)
   (f) \( D_y \left[ \frac{2a^2}{9y^3} \right] \) (\( a \) is a constant)

3. Determine the value of \( k \) for which the gradient of the tangent to \( g(x) = 3x^2 + kx + 4 \) at the point where \( x = 2 \) is 8.

4. Determine the derivative of the following using first principles:
   (a) \( f(x) = \frac{1}{x} \)
   (b) \( g(x) = -2x^3 \)
   (c) \( D_x \left[ \frac{x^3 - 27}{x - 3} \right] \) at \( x = 1 \) (Hint: First simplify the expression)

5. Find the equation of the tangent to \( f(x) = 3x^2 - 5x + 1 \) which is parallel to the line \( y - 7x + 4 = 0 \).

6. Sketch the graph of \( f(x) = x^3 - 3x^2 + 4 \)
   Indicate the coordinates of the stationary points, intercepts with the axes and any points of inflection.

7. Given: \( f(x) = x^3 + x^2 - 5x + 3 \)
   (a) Calculate the \( x \)- and \( y \)-intercepts of \( f \).
   (b) Determine the coordinates of the turning points.
   (c) Determine the \( x \)-value of the point of inflection.
   (d) Hence, sketch the graph of \( f \). Clearly show intercepts with the axes as well as the turning points.

8. Consider the function \( f : x \rightarrow -x^3 + 3x^2 + 9x + 5 \)
   (a) Draw a neat sketch graph of \( f \). Clearly indicate the coordinates of the intercepts with the axes as well as the stationary points and point of inflection.
   (b) Determine graphically the value(s) of \( k \) for which \( x^3 - 3x^2 - 9x + k = 0 \) has one real root.

9. The turning points of the graph of \( f(x) = -2x^3 + ax^2 + bx + c \) are \( (2 ; -9) \) and \( (5 ; 18) \). Determine the value of \( a, b \) and \( c \).

10. If \( f(x) = x^3 + 3x^2 - 9x \), determine the values of \( k \) for which \( y = -9x + k \) will be a tangent to the curve of \( f \).
11. The graph of \( f(x) = ax^3 + bx^2 + 24x - 20 \) is sketched alongside (not to scale). A(2 ; 0) is a local maximum turning point. B is the y-intercept of \( f \) and C is an x-intercept of \( f \). D is a local minimum turning point. E is the point of inflection on the graph of \( f \). Determine:
(a) the values of \( a \) and \( b \).
(b) the coordinates of B.
(c) the length of AC.
(d) the coordinates of D.
(e) the coordinates of E.

12. The graph of \( y = g'(x) \) is sketched alongside. This graph represents the derivative graph of a quadratic function \( g \).
(a) Determine the values of \( x \) for which the graph of \( g \) decreases.
(b) Write down the \( x \)-coordinate of the turning point of \( g \).
(c) Show that the turning point of \( g \) is a maximum.

13. The graphs of \( y = f'(x) = ax^2 + bx + c \) and \( g(x) = 2x - 4 \) are sketched below. The graph of \( y = f'(x) = ax^2 + bx + c \) is the derivative graph of a cubic function \( f \). The graph of \( f' \) cuts the x-axis at (1 ; 0) and (3 ; 0) and the y-axis at (0 ; 3) . B is the x-intercept of \( g \) and C is the turning point of \( f' \). BC \parallel y-axis.
(a) Write down the coordinates of A.
(b) Determine the equation of the graph of \( f' \) in the form \( y = ax^2 + bx + c \)
(c) Write down the \( x \)-coordinates of the turning points of \( f \).
(d) Determine the values of \( x \) for which the graph of \( f \) is increasing.
(e) Determine the values of \( x \) for which the graph of \( y = f'(x) \) is decreasing.
(f) Write down the \( x \)-coordinate of the point of inflection of the graph of \( f \).
(g) Sketch the graph of \( f \) if it is further given that:
\[ f(0) = 0 \quad f(1) = \frac{4}{3} \quad f(3) = 0 \quad f(2) = \frac{2}{3} \]
14. A builder wishes to construct a steel window frame in the shape of a rectangle with a semi-circular part on top. The radius of the semi-circular part is \( r \) metres and the width of the rectangular part is \( h \) metres.
   (a) Write down, in terms of \( h \) and \( r \)
       (1) the steel perimeter (P) of the frame.
       (2) the area enclosed by the frame.
   (b) The area enclosed by the frame is to be 4 square metres.
       Show that the perimeter (P) is
       \[
       P = \left( \frac{\pi}{2} + 2 \right) r + \frac{4}{r}
       \]
   (c) If the steel for the frame costs R10 per metre, calculate the value of \( r \) for which the total cost of the steel will be a minimum.

15. The dimensions of a rectangular room are:
   length 6 metres, breadth 4 metres and height 3 metres.
   If the dimensions are altered so that the length is reduced by \( x \) metres, the breadth is increased by \( 2x \) metres and the height remains constant, calculate:
   (a) the volume of the air in the new room in terms of \( x \).
   (b) the value of \( x \) for which the volume is a maximum.
   (c) the maximum volume.

16. A scientist adds a bactericide into a culture of bacteria. The number of bacteria present \( t \) hours after the bactericide was introduced is given by the formula:
   \[
   B(t) = 1000 + 50t - 5t^2
   \]
   (a) How many bacteria were in the culture when the bactericide was introduced?
   (b) Calculate the rate of change of the number of bacteria with respect to time three hours after the bactericide was added.
   (c) When does the population of bacteria start to decrease?
   (d) When will the whole culture of bacteria be exterminated?

17. The depth (in metres) of water left in a dam \( t \) hours after a sluice gate has been opened to allow water to drain from the dam, is given by the formula:
   \[
   d = 28 - \frac{1}{9}t^2 - \frac{1}{27}t^3
   \]
   (a) Calculate the average rate at which the depth changes in the first three hours.
   (b) Determine the rate at which the depth changes after exactly two hours.

18. A stone is thrown vertically upwards and its height (in metres) above the ground at time \( t \) (in seconds) is given by:
   \[
   h(t) = 35 - 5t^2 + 30t
   \]
   (a) Determine its initial height above the ground.
   (b) Determine the initial speed with which it was thrown.
   (c) Determine the maximum height above the ground that the stone reached.
   (d) How fast was the stone travelling when it reached a height of 60 metres above the ground on the way down?
   (e) How fast was the stone travelling when it hit the ground?
SOME CHALLENGES

1. (a) If \( f(x) = \frac{16}{x} \), determine \( f'(4x) - 4f'(x) \).

(b) Find \( \frac{d}{dx} \left[ f(x) + 3g(x) \right] \) if \( f(x) = \left(\sqrt{x}\right)^3 + 7 \) and \( g'(x) = -7 \).

(c) The gradient of a function \( f(x) = ax^2 + bx + c \) is given by \( f'(x) = 3x + 4 \). If \( f(0) = 5 \), determine the function \( f(x) \).

(d) If \( y = 3x + 5 \), determine \( \frac{dy}{dx} + \frac{dx}{dy} \)

(e) Determine \( \frac{dy}{dx} \) if \( xy - 5 = \sqrt{x^3} \)

(f) The line \( g(x) = 5x + 1 \) is a tangent to the curve of a function \( f \) at the point where \( x = 2 \). Calculate the value of \( f(2) + f'(2) \).

2. Consider the following functions:
\( f(x) = x^3 + 3x^2 + 3x + 1 \) and \( g(x) = -5x^3 - 5x \)

(a) Show that the graph of \( f \) increases for all values of \( x \) provided that \( x \neq -1 \)

(b) Show that the graph of \( f \) has one stationary point at \( x = -1 \)

(c) Show that this stationary point is also a point of inflection.

(d) Draw a neat sketch graph of \( f \).

(e) Explain why it is not possible for any tangent drawn to the graph of \( f \) to have a negative gradient.

(f) Will the graph of \( g \) ever increase? Explain your answer.

(g) Explain if it possible to draw a tangent to the graph of \( g \) that has a positive gradient.

(h) Determine the coordinates of the point of inflection of \( g \).

3. Redraw the given diagram of \( y = f(x) \).
Now draw a possible graph of \( y = f'(x) \) on the same set of axes.

4. A is a town 30 km west of town B.
Two athletes start walking simultaneously from the two towns. The athlete who starts from town A, walks due east in the direction of B at a constant speed of 6 km/h and reaches point P after \( x \) hours. The athlete, who starts at B, walks due north in the direction of another town C at a constant speed of 8 km/h, and reaches point Q after \( x \) hours.
(a) Prove that the distance between the athletes after \( x \) hours is given by:
\[
PQ = \sqrt{100x^2 - 360x + 900}.
\]
(b) How long after they started walking were they nearest to each other?
(c) What was this minimum distance between them?

5. The equation of ED is \( y = -2x + 10 \)
\((x > 0, \ y > 0)\) and \( B(x; y) \) is any point
on ED. A is the point \((0; 8)\) and \( BC \perp OX \).
Determine the coordinates of B for the
shaded area AOCB to be a maximum.

6. An equilateral triangle (ABC) of side \( a \) units is to
have a rectangle DEFG inscribed in it. \( AE = FB = x \).

(a) Prove that the area of the rectangle
is given by: \( A(x) = \sqrt{3}a(a - 2x) \)
(b) Determine, in terms of \( a \), the value of \( x \)
for which the area of the rectangle is a maximum.

7. Triangular off-cuts of steel plating are
in the form of a right-angled
triangles ABC, where
\( AB = 200mm \) and \( BC = 400mm \).
Rectangular metal plates FBDE are
cut from these triangles.
(a) If \( FE = y \) and \( DE = x \),
prove that \( y + 2x = 400 \).
(b) Hence determine an expression
for the area of the rectangular metal
plates in terms of \( x \).
(c) Determine the dimensions of these plates so that their areas will
have a maximum value.
CHAPTER 7 – ANALYTICAL GEOMETRY

SUMMARY OF ANALYTICAL GEOMETRY THEORY (GRADE 10-11)

1. Length of line AB:
   \[ AB^2 = (x_B - x_A)^2 + (y_B - y_A)^2 \]  
   or
   \[ AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} \]  
   with \( A(x_A; y_A) \) and \( B(x_B; y_B) \)

2. Formula for any point \( M \), the midpoint of a line segment \( AB \):
   \[ M \left( \frac{x_M + x_A}{2}, \frac{y_M + y_A}{2} \right) \]  
   with \( A(x_A; y_A) \) and \( B(x_B; y_B) \)

3. The gradient of line \( AB \):
   \[ m_{AB} = \frac{y_B - y_A}{x_B - x_A} \]  
   with \( A(x_A; y_A) \) and \( B(x_B; y_B) \)

4. If \( \theta \) is the angle of inclination of line \( OR \) then: \( \tan \theta = \text{gradient}_{OR} \)

5. (a) Parallel lines: Non-vertical lines with gradients \( m_1 \) and \( m_2 \) are parallel if their gradients are equal. \( m_1 = m_2 \)
   (b) Perpendicular lines: Non-vertical lines with gradients \( m_1 \) and \( m_2 \) respectively are perpendicular if \( m_1m_2 = -1 \)
   (c) If \( A, B \) and \( C \) are collinear then \( \text{gradient}_{AB} = \text{gradient}_{BC} = \text{gradient}_{AC} \)

6. Equation of a straight line
   (a) \( y = mx + c \) \( m \) is the gradient and \( c \) is the \( y \)-intercept
      (Gradient-intercept form).
   (b) \( y - y_1 = m(x - x_1) \) straight line with gradient \( m \) passing through
      the point \((x_1; y_1)\)
   (c) Vertical line: \( x = \text{number} \) gradient is undefined
   (d) Horizontal line: \( y = \text{number} \) gradient is zero

REVISION EXERCISE

1. In the diagram, \( A(-2; -3) \), \( B(4; -5) \) and \( C(2; 1) \) are vertices of a triangle in a Cartesian plane. \( CG \) is a line such that \( AG = GB \) and \( AE \perp BC \).
(a) Calculate the coordinates of G, the midpoint of AB.
(b) Calculate the length of CG. Leave your answer in surd form.
(c) Calculate the value of \( \theta \), the angle of inclination of BC, rounded off to one decimal digit.
(d) Determine the equation of the straight line FA, which is parallel to the line BC.
(e) Determine the equation of AE.
(f) Calculate the \( y \)-intercept of AE.

2. Consider the following points on a Cartesian plane: A(1; 2), B(3; 1), C(\(-3; k\)) and D(2; \(-3\)). Determine the value(s) of \( k \) if:
(a) \((-1; 3)\) is the midpoint of AC.
(b) AB is parallel to CD.
(c) AB \( \perp \) CD.
(d) A, B and C are collinear.
(e) CD = \(5\sqrt{2}\)

3. A(1; 4), B(\(-2; 2\)), C(4; 1) and M(\(x; y\)) are the points on the Cartesian plane.
Use analytical methods to determine the coordinates of M so that ABCM, in this order, is a parallelogram.

4. Determine:
(a) the size of \( \alpha \)
(b) the size of \( \beta \)
(c) the size of \( \theta \)

CIRCLES

A circle is a set of points which are equidistant from a fixed point, the centre of the circle. The distance from the centre to any point on the circle is the length of the radius while the distance around the whole circle is the circumference or the perimeter of the circle.

CIRCLE WITH CENTRE THE ORIGIN

Suppose \( P(x; y) \) is always \( r \) units from the origin, (0; 0).
(This implies \( P(x; y) \) lies anywhere on the circle)
From the distance formula we know that:
\[
OP^2 = (x - 0)^2 + (y - 0)^2
\]
\[\therefore OP = x^2 + y^2\]
But \( OP = r \), and therefore \( OP^2 = r^2\)
\[\therefore x^2 + y^2 = r^2\]
The equation of the circle centre (0; 0) is therefore \( x^2 + y^2 = r^2 \)
EXAMPLE 1

(a) Determine the equation of the circle with centre the origin and radius 5.

\[ x^2 + y^2 = r^2 \]  (equation of a circle with centre the origin)
\[ \therefore x^2 + y^2 = 5^2 \]  (substitute \( r = 5 \))
\[ \therefore x^2 + y^2 = 25 \]

(b) Determine the equation of the circle with centre the origin passing through the point \((-2; 4)\).

Sketch the graph first.
\[ x^2 + y^2 = r^2 \]  (state the equation)
\[ \therefore (-2)^2 + (4)^2 = r^2 \]  (substitute the point \((-2; 4)\) that lies on the circle)
\[ \therefore 4 + 16 = r^2 \]
\[ \therefore r^2 = 20 \]
Therefore the equation of the circle is: \( x^2 + y^2 = 20 \)

EXAMPLE 2

\( (b; -\sqrt{8}) \) is a point on the circle with equation \( x^2 + y^2 = 17 \).

(a) Determine the possible values of \( b \).
(b) Determine the \( x \) and \( y \)-intercepts of the circle.
(c) Find one other point with integer coordinates that lies on the circle.

Solutions

(a) \( x^2 + y^2 = 17 \)  (state the equation of graph)
\[ \therefore b^2 + (-\sqrt{8})^2 = 17 \]  (substitute point \((b; -\sqrt{8})\))
\[ \therefore b^2 + 8 = 17 \]
\[ \therefore b^2 = 9 \]
\[ \therefore b = \pm \sqrt{9} \]
\[ \therefore b = 3 \text{ or } b = -3 \]

(b) Finding the \( x \)-intercepts: Let \( y = 0 \):
\[ \therefore x^2 + 0^2 = 17 \]
\[ \therefore x^2 = 17 \]
\[ \therefore x = \pm \sqrt{17} \]
\[ \therefore x = \sqrt{17} \text{ or } x = -\sqrt{17} \]
Finding the \( y \)-intercepts: Let \( x = 0 \):
\[ \therefore 0^2 + y^2 = 17 \]
\[ \therefore y^2 = 17 \]
\[ \therefore y = \pm \sqrt{17} \]
\[ \therefore y = \sqrt{17} \text{ or } y = -\sqrt{17} \]

(c) This will be done by inspection.
Consider any integer \( x \)-coordinate that lies within the domain \([-\sqrt{17}; \sqrt{17}]\).
\[
\sqrt{17} = 4.1231\ldots.
\]
\[
(1) + y^2 = 17 \quad \text{(substitute } x = 1 \text{ (part of the domain))}
\]
\[
\therefore y^2 = 16
\]
\[
\therefore y = \pm \sqrt{16} \quad \therefore y = 4 \text{ or } y = -4
\]
\[
(1; 4) \text{ or } (-1; 4) \text{ are points with integer coordinates that lie on the circle.}
\]
If the first value you substituted didn’t produce an integer value then you continue substituting \(x\)-values in the domain until it produces an integer value.

**EXERCISE 1**

1. Determine the equation of each of the following circles

   (a) \[x^2 + y^2 = 25\]

   (b) \[x^2 + y^2 = 8\]

   (c) \[2x^2 + 2y^2 = 32\]

   (d) \[x^2 + y^2 - 9 = 0\]

   (e) \[x^2 + y^2 = 1\]

2. Draw sketch graphs of the following circles and show at least one point other than the \(x\) or \(y\)-intercept on the circle.

   (a) \[x^2 + y^2 = 25\]

   (b) \[x^2 + y^2 = 8\]

   (c) \[2x^2 + 2y^2 = 32\]

   (d) \[x^2 + y^2 - 9 = 0\]

   (e) \[x^2 + y^2 = 1\]

3. Determine the equation of the circle with centre at the origin, and:

   (a) a radius of 4 units

   (b) a radius of \(\sqrt{10}\)

   (c) passing through the point \((-2; -4)\)

   (d) passing through the point \((\sqrt{3}; 1)\)

   (e) passing through the point \((5; -1)\)

   (f) a radius of \(2\sqrt{5}\) units
4. (a) Point \((-2; b)\) lies on the circle \(x^2 + y^2 = 13\). Determine the values of \(b\).
(b) Point \((a; \sqrt{5})\) lies on the circle \(x^2 + y^2 = 21\). Determine the values of \(a\).

5. \(P(-5; 12)\) lies on the circle with centre the origin:
(a) Determine the equation of the circle
(b) Determine the coordinates of \(Q\) if \(PQ\) is a diameter.
(c) Show that the point \(M(0; -13)\) lies on the circle
(d) Show that \(PMQ = 90^\circ\).

**Circles not centred at the origin**

Let \(P(x; y)\) be any point on the circle with centre \(M(a; b)\) and radius \(r\).

From the distance formula we know that:
\[MP^2 = (x-a)^2 + (y-b)^2\]

But \(MP = r\), and therefore \(MP^2 = r^2\)
\[\therefore (x-a)^2 + (y-b)^2 = r^2\]

The equation of a circle with centre \((a; b)\) is:
\[(x-a)^2 + (y-b)^2 = r^2\]

Another standard form of a circle’s equation with any centre is
\[x^2 + dx + y^2 + ey + f = 0\]. You can manipulate this equation to
\[(x-a)^2 + (y-b)^2 = r^2\] so that the centre and radius of the circle can be determined.

**Example 3**

Determine the co-ordinates of the centre and the length of the radius of each of the following circles:
(a) \((x - 4)^2 + (y + 3)^2 = 18\) \(\quad\) (b) \(x^2 + y^2 - 6x + 5y - 4 = 0\)
(c) \(x^2 + 2x + y^2 - 10y + 1 = 0\)

**Solutions**

(a) \((x - 4)^2 + (y + 3)^2 = 18\)
\[\therefore (x - 4)^2 + (y - (-3))^2 = 18\]
Centre: \((4; -3)\) with \(r^2 = 18\)
\[\therefore r = \sqrt{18} = 3\sqrt{2}\]

(b) To find the centre and the radius of any circle you have to manipulate the above equation to the form \((x-a)^2 + (y-b)^2 = r^2\)
Firstly, move the constant over and group all the terms with \(x\)’s and \(y\)’s together.
\[\therefore x^2 - 6x + y^2 + 5y = 4\]
Secondly, complete the square for the expressions in $x$ and $y$ separately.

$x^2 - 6x + y^2 + 5y = 4$

\[
\left(\frac{1}{2}\text{ coefficient of } x\right)^2 = \left(\frac{1}{2}(-6)\right)^2 = (-3)^2 \quad \text{and}
\]

\[
\left(\frac{1}{2}\text{ coefficient of } y\right)^2 = \left(\frac{1}{2}(5)\right)^2 = \left(\frac{5}{2}\right)^2
\]

Add the above constants to both sides of the equation:

\[
\therefore x^2 - 6x + (-3)^2 + y^2 + 5y + \left(\frac{5}{2}\right)^2 = 4 + (-3)^2 + \left(\frac{5}{2}\right)^2
\]

\[
\therefore (x - 3)^2 + \left(y + \frac{5}{2}\right)^2 = 19 \frac{1}{4}
\]

Centre: \(3; -\frac{5}{2}\) with \(r^2 = 19 \frac{1}{4}\)

\[
:\therefore r = \sqrt{19 \frac{1}{4}} = \frac{\sqrt{77}}{2}
\]

(c) \(x^2 + 2x + y^2 - 10y = -1\)

\[
\therefore x^2 + 2x + \left(\frac{2}{2}\right)^2 + y^2 - 10y + \left(\frac{-10}{2}\right)^2 = -1 + \left(\frac{2}{2}\right)^2 + \left(\frac{-10}{2}\right)^2
\]

\[
\therefore x^2 + 2x + 1^2 + y^2 - 10y + (-5)^2 = -1 + 1^2 + (-5)^2
\]

\[
\therefore (x + 1)^2 + (y - 5)^2 = 25
\]

Centre: \((-1; 5)\) with \(r^2 = 25\)

\[
\therefore r = 5
\]

**EXAMPLE 4**  (Finding the equation)

(a) Write down the equation of the circle with centre \((-1; 3)\) and radius \(2\sqrt{3}\)

(b) Determine the equation of the circle with centre \(M(-1; 3)\) and \(A(-4; -1)\) a point on the circle.

**Solutions**

(a) \((x - a)^2 + (y - b)^2 = r^2\)

Substitute the centre \((-1; 3)\) and \(r = 2\sqrt{3}\)

\[
\therefore (x - (-1))^2 + (y - 3)^2 = (2\sqrt{3})^2
\]

\[
\therefore (x + 1)^2 + (y - 3)^2 = 12
\]
EXERCISE 2

1. Determine the coordinates of the centre and the length of the radius of each of the following circles:
   (a) \((x - 4)^2 + (y + 3)^2 = 25\)
   (b) \(x^2 + (y - 2)^2 = 10\)
   (c) \(x^2 + 2x + y^2 = 0\)
   (d) \(x^2 - 8x + y^2 + 6y = 2\)
   (e) \(x^2 - 5x + y^2 + 6y = 9\)
   (f) \(x^2 + x + y^2 - 4y - 4 = 0\)
   (g) \(x^2 + y^2 + 10x - 3y + 1 = 0\)
   (h) \(x^2 + y^2 - 2x - y - 12 = 0\)
   (i) \(2x^2 + 2y^2 - 6x - 12y = 3\)

2. Determine the equation of the circle in each case:
   (a) with centre \((-2; -1)\) and radius 4
   (b) with centre \((3; -3)\) and radius \(3\sqrt{5}\)
   (c) with centre \(M(-3; 1)\) and \(A(2; -2)\) a point on the circle.
   (d) with centre \((1; -1)\) and \((-3; 4)\) a point on the circle.

3. The equation of a circle with radius \(3\sqrt{2}\) units is \(x^2 - 6x + y^2 + 2y - m = 0\).
   (a) Determine the coordinates of the centre of the circle.
   (b) Find the value of \(m\).

4. Make rough sketch graphs of the circles below.
   (a) Determine the equation of the circle with BC as diameter if B and C are the points \((2; -3)\) and \((6; -1)\) respectively.
   (b) Determine the equation of the circle with centre \((4; b)\), \(b > 0\) which cuts the y-axis at \((0; 1)\) and the x-axis at \((7; 0)\).

5. A circle with centre \(M(-5; 2)\) touches the y-axis at E and passes through F.
   (a) Write down the coordinates of E and F.
   (b) Determine the equation of the circle.
   (c) Determine whether the point \((-3; 6)\) lies inside or outside the circle.
EXAMPLE 5

Show that the straight line \( y = -x - 4 \) is a tangent to the circle \((x-1)^2 + (y+1)^2 = 8\).

Solution

How does one show that a line is a tangent to a circle? Remember that a tangent to a circle intersects (touches) the circle at one point only. Therefore it will be necessary to show that the two graphs intersect at one point only.

\[ y = -x - 4 \quad \text{.... A} \]
\[ (x-1)^2 + (y+1)^2 = 8 \quad \text{.... B} \]

Substitute A in B:

\[ (x-1)^2 + ((-x-4)+1)^2 = 8 \]
\[ (x-1)^2 + (-x-3)^2 = 8 \]
\[ x^2 - 2x + 1 + x^2 + 6x + 9 = 8 \]
\[ 2x^2 + 4x + 2 = 0 \]
\[ x^2 + 2x + 1 = 0 \]
\[ (x+1)(x+1) = 0 \]
\[ x = -1 \]

Substitute \( x = -1 \) into A

\[ y = -(-1) - 4 \]
\[ y = 1 - 4 = -3 \]

Therefore straight line \( y = -x - 4 \) is a tangent to the circle \((x-1)^2 + (y-1)^2 = 8\) because the straight line intersects the circle in one point only which is \((-1; -3)\).
THE EQUATION OF A TANGENT TO A CIRCLE

EXAMPLE 6

Find the equation of the tangent to the circle \( x^2 - 6x + y^2 + 2y = 0 \) at \( (4; 2) \)

**Solution**

Firstly you have to find the centre of the circle:

\[
\begin{align*}
  x^2 - 6x + y^2 + 2y &= 0 \\
  \therefore x^2 - 6x + \left(\frac{-6}{2}\right)^2 + y^2 + 2y + \left(\frac{2}{2}\right)^2 &= 0 + \left(\frac{-6}{2}\right)^2 + \left(\frac{2}{2}\right)^2 \\
  \therefore x^2 - 6x + (-3)^2 + y^2 + 2y + (1)^2 &= 0 + (-3)^2 + (1)^2 \\
  \therefore (x - 3)^2 + (y + 1)^2 &= 10.
\end{align*}
\]

The centre of the circle is \( (3; -1) \)

Secondly you have to find the gradient of the radius through the centre \( (3; -1) \) and the point \( (4; 2) \) where the tangent touches the circle.

Gradient of radius \( \frac{2 - (-1)}{4 - 3} = 3 \)

\[ \therefore m_{rad} = 3 \]

Now it is possible to find the gradient of the tangent:

\( \text{gradient of radius} \times \text{gradient of tangent} = -1 \) \( \text{(radius } \perp \text{ tangent)} \)

\[ \therefore m_{tan} = -\frac{1}{3} \]

Thirdly you substitute the point that lies on the tangent and the gradient of the tangent, into the standard form of a straight line: \( y - y_i = m(x - x_i) \)

Equation of tangent:

\[
\begin{align*}
  y - y_i &= m_{tan}(x - x_i) \\
  \therefore y - 2 &= -\frac{1}{3}(x - 4) \quad \text{(substitute } m_{tan} = -\frac{1}{3} \text{ and } (4; 2)) \\
  \therefore y - 2 &= -\frac{1}{3}x + \frac{4}{3} \\
  \therefore y &= -\frac{1}{3}x + \frac{10}{3}
\end{align*}
\]

The equation of the tangent to the circle at the point \( (4; 2) \) is \( y = -\frac{1}{3}x + \frac{10}{3} \)
EXAMPLE 7

The line defined by \( y + 2x = 11 \) is a tangent to a circle with centre \( P(1; -1) \) at the point \( Q(x; y) \).

(a) Determine the equation of the radius \( PQ \) and hence the coordinates of \( Q \).

(b) Determine the equation of the circle through \( Q \) with centre \( P \).

Solutions

(a) To determine the equation of any straight line you need to find the gradient of that line and a point on that line to substitute into the equation.

\[ y - y_1 = m(x - x_1) \]

Let us write the equation of the tangent in the form \( y = mx + c \):

\[ y + 2x = 11 \]

\[ \therefore y = -2x + 11 \]

\[ \therefore m_{\text{tan}} = -2 \]

We know that \( m_{\text{tan}} \times m_{\text{rad}} = -1 \) (radius \( \perp \) tangent)

\[ \therefore m_{\text{rad}} = \frac{1}{2} \]

Therefore the equation of the radius \( PQ \) is:

\[ y - y_1 = m_{\text{rad}}(x - x_1) \]

\[ \therefore y - y_1 = \frac{1}{2}(x - x_1) \]

Substitute the point \( P(1; -1) \) that lies on the radius:

\[ \therefore y - (-1) = \frac{1}{2}(x - 1) \]

\[ \therefore y + 1 = \frac{1}{2}x - \frac{1}{2} \]

\[ \therefore y = \frac{1}{2}x - \frac{3}{2} \]

The equation of the radius \( PQ \) is: \( y = \frac{1}{2}x - \frac{3}{2} \)

The point \( Q(x; y) \) is where the radius and the tangent meet (intersect).

Therefore we have to solve these two equations simultaneously.

\[ y + 2x = 11 \quad \text{...A} \quad \text{and} \quad y = \frac{1}{2}x - \frac{3}{2} \quad \text{...B} \]

Substitute B into A

\[ \therefore \frac{1}{2}x - \frac{3}{2} + 2x = 11 \quad \text{LCD: 2} \]

\[ \therefore \frac{5x}{2} = 22 \]

\[ \therefore 5x = 25 \]

\[ \therefore x = 5 \]
Substitute $x = 5$ into B

$$y = \frac{1}{2}(5) - \frac{3}{2}$$

$$\therefore y = 1$$

The coordinates of Q are: (5; 1)

(b) \( (x-a)^2 + (y-b)^2 = r^2 \)

PQ = \( r \) and therefore \( PQ^2 = r^2 \) and:

\[
PQ^2 = (1-5)^2 + (-1-1)^2
\]

\[
\therefore r^2 = 20
\]

The equation of the circle is: \( (x-1)^2 + (y+1)^2 = 20 \)

**EXAMPLE 8**

Consider a circle with equation \( x^2 - 2x + (y + 3)^2 = 13 \). A tangent is drawn from (2; 4) to point A\( (x; y) \) on the circle. Determine the length of the tangent TA.

![Diagram of a circle and tangent](image)

**Solution**

In the diagram, TA \( \perp MA \) since the radius is perpendicular to the tangent at the point of contact.

From Pythagoras, \( TA^2 = TM^2 - AM^2 \)

The coordinates of centre M are required in order to determine the length of TM, and hence the length of TA.

\[x^2 - 2x + (y + 3)^2 = 13\]

\[
\therefore x^2 - 2x + \left(\frac{-2}{2}\right)^2 + (y + 3)^2 = 13 + \left(\frac{-2}{2}\right)^2
\]
\[ x^2 - 2x + (-1)^2 + (y + 3)^2 = 13 + (-1)^2 \]
\[ (x - 1)^2 + (y + 3)^2 = 14 \]

The centre is \( M(1; -3) \) and radius \( AM = \sqrt{14} \)

\[ TM^2 = (1 - 2)^2 + (-3 - 4)^2 = 50 \]
\[ TA^2 = TM^2 - AM^2 \]
\[ TA^2 = 50 - (\sqrt{14})^2 \]
\[ \therefore TA^2 = 36 \]
\[ \therefore TA = 6 \]

**EXERCISE 3**

1. Determine the equation of the tangent which touches the circle:
   (a) \( x^2 + y^2 = 13 \) at the point \((3; -2)\)
   (b) \((x - 2)^2 + (y + 3)^2 = 17\) at the point \((1; 1)\)
   (c) \((x + 4)^2 + (y - 1)^2 = 20\) at the point \((-2; -3)\)
   (d) \((x^2 - 2x + y^2 + 4y - 5 = 0\) at the point \((-2; -1)\)
   (e) \((x^2 + 6x + y^2 + 12y = 13\) at the point \((-6; 1)\)

2. Find the equations of the tangents to \((x - 2)^2 + (y + 3)^2 = 16\) which are:
   (a) parallel to the y-axis.
   (b) parallel to the x-axis.

3. The straight line \( y = x + 2 \) cuts the circle \( x^2 + y^2 = 20 \) at A and B.
   (a) Determine the coordinates of A and B.
   (b) Determine the length of the chord AB.
   (c) Determine the coordinates of M, the midpoint of the chord AB.
   (d) Show that \( OM \perp AB \) if O is the origin.
   (e) Determine the equations of the tangents to the circle at A and B.
   (f) Determine the coordinates of C, the point of intersection of the tangents found in (e).

4. The straight line \( y = -2x + c \) is a tangent to the circle \((x + 1)^2 + y^2 = 20\) at A\((x; y)\).
   (a) Determine the equation of the radius through A.
   (b) Determine the coordinates of A and hence the value of \( c \).

5. A circle with centre M is given. BE and EL are tangents to the circle at D and N respectively. The equation of the circle is: \( x^2 - 8x + y^2 - 4y + 15 = 0 \)
   The equation of the tangent BE is: \( y = \frac{1}{2}x + c \)
   (a) Determine the centre M and the radius of the circle
(b) Determine the equation of the radius MD and hence show that the coordinates of D are (3 ; 4).

(c) If the coordinates of L are (9 ; 0) calculate the length of LN.

6. A circle with centre P(−4 ; 2) has the points O(0 ; 0) and N(−2 ; b) on the circumference. The tangents at O and N meet at R.

Determine:
(a) the equation of the circle.
(b) the value of b.
(c) the equation of OR.
(d) the coordinates of R.

REVISION EXERCISE  (GRADE 11 AND GRADE 12 WORK)

1. P(3 ; 1) lies on a circle with centre (−1 ; 4) :
   (a) Determine the equation of the circle
   (b) Determine the coordinates of Q if PQ is a diameter.
   (c) Show that the point M(−4 ; 8) lies on the circle.
   (d) Show that PMQ = 90°.

2. Determine the equation of the circle with BC as diameter if B and C are the points (−3 ; 2) and (1 ; 6) respectively.

3. Determine the co-ordinates of the centre and the length of the radius of each of the following circles:
   (a) $x^2 + 3x − 6y + y^2 = 9$
   (b) $x^2 + y^2 − 6y − 3 = 0$

4. Determine the equation of the tangent which touches the circle:
   (a) $(x − 1)^2 + y^2 = 13$ at the point (−3 ; 2)
   (b) $x^2 − 10x + y^2 + 3y − 10 = 0$ at the point (1 ; 6)

5. Determine the value of p if A(−3; p) is equidistant from the points C(7;−1) and D(4;−4).

6. Determine the values of x and y if (−2 ; y) is the midpoint of the line between the points (4 ; 3) and (x ; 7).
7. Refer to the diagram alongside. \( \triangle MNP \) is given with \( M(2;4) \), \( N(-3;-2) \) and \( P(-9;-1) \). Determine the size of angle \( M \).

8. Given: ABCD is a rhombus with \( A(-1;-2) \) and \( C(3;4) \).
   (a) Determine the equation of AC.
   (b) Determine the equation of BD.
   (c) If \( D \) is the point \( D(6;y) \), determine the value of \( y \) and hence the coordinates of B.

9. The equation of a straight line AB is \( y = -\frac{3}{4}x + 1 \). The equation of a straight line CD is \( py = 3x - 2 \). Find the value of \( p \) in each case, if:
   (a) AB\|CD
   (b) AB cuts CD at \( (2;1) \)
   (c) the angle of inclination of CD is 120°

10. Given: \( x^2 + y^2 + 8x - 12y + 27 = 0 \)
    (a) Calculate the centre and the radius of the circle.
    (b) Calculate \( t \) if a second circle with the equation \( x^2 + y^2 + 8x - 12y + t = 0 \)
        has a radius twice that of the original.

11. The straight line passing through \( P(1;3) \) has an angle of inclination of 45° and cuts the circle \( x^2 + y^2 = 20 \) at H and G. Show that the co-ordinates of H and G are \( (-4;-2) \) and \( (2;4) \).
SOME CHALLENGES

1. Consider the diagram below and then determine the area of \( \triangle PQC \).

   ![Diagram of \( \triangle PQC \)]

2. Amy stands by a pool. She throws a pebble in the water and watches the ripples that form. The ripples are represented by 6 circles, all with the same centre.

3. Show that the following two circles touch each other at one point:
   \[ x^2 - 4x + y^2 + 4y + 3 = 0 \quad \text{and} \quad (x - 4)^2 + (y - 2)^2 = 5 \]

4. Determine the equation of the circle going through the points \( A(4; 4), B(3; 1) \) and \( C(-4; 8) \).

The equation of the smallest circle, \( w_1 \), is \( x^2 + y^2 + 4x - 6y + k = 0 \) and the equation of the circle, \( w_6 \), is \( x^2 + y^2 + 4x - 6y = 87 \). If the circle \( w_6 \) has a radius which is 5 times that of \( w_1 \), calculate the value of \( k \).
5. Refer to the diagram below. O is the centre of the circle, E lies on the y-axis and $G(3; -4)$ lies on the circle. EF and FG are tangents to the circle.

Determine the coordinates of F.

6. A circle with centre M passes through the points $A(0; 7)$, $D(4; -2)$ and $F(x; y)$. Determine the coordinates of F in surd form. Show all working out.
CHAPTER 8 – EUCLIDEAN GEOMETRY

REVISION OF EARLIER CONCEPTS (GRADE 8-11)

LINES AND ANGLES

Adjacent supplementary angles
In the diagram, \( \hat{B}_1 + \hat{B}_2 = 180^\circ \)

Angles round a point
In the diagram, \( a + b + c = 360^\circ \)

Vertically opposite angles
Vertically opposite angles are equal.

Corresponding angles
If AB\( \parallel \)CD, then the corresponding angles are equal.

Alternate angles
If AB\( \parallel \)CD, then the alternate angles are equal.

Co-interior angles
If AB\( \parallel \)CD, then the co-interior angles add up to \( 180^\circ \), i.e. \( x + y = 180^\circ \)

TRIANGLES

Scalene Triangle
No sides are equal in length

Isosceles Triangle
Two sides are equal
Base angles are equal
**Equilateral Triangle**
All three sides are equal
All three interior angles are equal

![Equilateral Triangle]

**Right-angled triangle**
One interior angle is $90^\circ$

![Right-angled triangle]

**Sum of the angles of a triangle**

\[ a + b + c = 180^\circ \]

**Exterior angle of a triangle**

\[ c = a + b \]

**The Theorem of Pythagoras**

\[ AC^2 = AB^2 + BC^2 \]

or

\[ AB^2 = AC^2 - BC^2 \]

or

\[ BC^2 = AC^2 - AB^2 \]

**Congruency of triangles (four conditions)**

**Condition 1**
Two triangles are congruent if three sides of one triangle are equal in length to the three sides of the other triangle.

**Condition 2**
Two triangles are congruent if two sides and the included angle are equal to two sides and the included angle of the other triangle.

**Condition 3**
Two triangles are congruent if two angles and one side of a triangle are equal to two angles and a corresponding side of the other triangle.
**Condition 4**
Two right-angled triangles are congruent if the hypotenuse and a side of the one triangle is equal to the hypotenuse and a side of the other triangle.

**The Midpoint Theorem**

If \( AD = DB \) and \( AE = EC \), then \( DE \parallel BC \) and \( DE = \frac{1}{2} BC \).

If \( AD = DB \) and \( DE \parallel BC \), then \( AE = EC \) and \( DE = \frac{1}{2} BC \).

**PROPERTIES OF QUADRILATERALS**

**Parallelograms**
If \( ABCD \) is a parallelogram, you may assume the following properties:

\[
\begin{align*}
\text{AD} & \parallel \text{BC} ; \text{AB} \parallel \text{DC} \\
\text{AD} & = \text{BC} ; \text{AB} = \text{DC} \\
\text{AE} & = \text{EC} ; \text{BE} = \text{ED} \\
\hat{D_1} & = \hat{B_2} ; \hat{D_2} = \hat{B_1} ; \hat{C_1} = \hat{A_2} ; \hat{C_2} = \hat{A_1} \\
\hat{A} & = \hat{C} ; \hat{B} = \hat{D}
\end{align*}
\]

In order to prove that a quadrilateral is a parallelogram, you will need to prove at least one of the following:

\[
\begin{align*}
\text{AD} & \parallel \text{BC} \text{ and } \text{AB} \parallel \text{DC} \quad \text{Opp sides } \parallel \\
\text{AD} & = \text{BC} \text{ and } \text{AB} = \text{DC} \quad \text{Opp sides } = \\
\text{AE} & = \text{EC} \text{ and } \text{BE} = \text{ED} \quad \text{Diagonals bisect} \\
\hat{A} & = \hat{C} \text{ and } \hat{B} = \hat{D} \quad \text{Opp angles } = \\
\text{AB} & \parallel \text{DC} \text{ and } \text{AB} = \text{DC} \quad \text{One pair opp sides } = \text{ and } \parallel \\
\text{AD} & \parallel \text{BC} \text{ and } \text{AD} = \text{BC} \quad \text{One pair opp sides } = \text{ and } \parallel
\end{align*}
\]
**Rectangle**
If ABCD is a rectangle, you may assume the following properties:

![Rectangle Diagram](image)

In order to prove that a quadrilateral is a rectangle, you will need to prove one of the following:

(a) The quadrilateral is a parallelogram with at least one interior angle equal to $90^\circ$.
(b) The diagonals of the quadrilateral are equal in length and bisect each other.

**Rhombus**
If ABCD is a rhombus, you may assume the following properties:

![Rhombus Diagram](image)

In order to prove that a quadrilateral is a rhombus, you will need to prove one of the following:

(a) The quadrilateral is a parallelogram with a pair of adjacent sides equal.
(b) The quadrilateral is a parallelogram in which the diagonals bisect at right angles.

**Square**
If ABCD is a square, you may assume the following properties:

![Square Diagram](image)

In order to prove that a quadrilateral is a square, you will need to prove one of the following:
(a) The quadrilateral is a parallelogram with an interior right angle and a pair of adjacent sides equal.
(b) The quadrilateral is a rhombus with an interior right angle.
(c) The quadrilateral is a rhombus with equal diagonals.

**Trapezium**
If ABCD is a trapezium, you may assume the following properties:

\[ \text{AD} \parallel \text{BC} \]
\[ \hat{A}_2 = \hat{C}_2 ; \hat{D}_1 = \hat{B}_2 \]

In order to prove that a quadrilateral is a trapezium, you will need to prove that \( \text{AD} \parallel \text{BC} \).

**Kite**
If ABCD is a kite, you may assume the following properties:

\[ \text{AB} = \text{AD} \]
\[ \text{BC} = \text{DC} \]
\[ \text{BE} = \text{ED} \]
\[ \hat{A}_1 = \hat{A}_2 \]
\[ \hat{C}_1 = \hat{C}_2 \]
\[ \hat{B} = \hat{D} \]
\[ \hat{E}_2 = 90^\circ \]
\[ \text{AC} \perp \text{BD} \]

In order to prove that a quadrilateral is a kite, you will need to prove that the pairs of adjacent sides are equal in length.

**CIRCLE GEOMETRY**

**Radius:**
A line from the centre to any point on the circumference of the circle.

**Diameter:**
A line passing through the centre of the circle. It is double the length of the radius.

**Chord:**
A line with end-points on the circumference.

**Tangent:**
A line touching the circle at only one point.

**Secant:**
A line passing through two points on the circle.
1. **When parallel lines are given**

![Diagram of parallel lines]

- **alt ∠s**
- **corr ∠s**
- **co-int ∠s suppl**

2. **How to prove that lines are parallel**

![Diagram of lines and angles]

Prove that $a = b$ or $a = d$ or $b + c = 180°$

3. **Angle or line bisectors**

- If BD bisects $\hat{ABC}$ then $\hat{B}_1 = \hat{B}_2$
- If CD bisects AB then $AE = EB$

4. **Triangle information**

- If $\hat{B} = \hat{C}$, then $AB = AC$.
- If $AB = AC$, then $\hat{B} = \hat{C}$.
- $\Delta ABC$ is **isosceles**

- $\hat{A} + \hat{B} + \hat{C}_2 = 180°$ (sum ∠s of $\Delta$)
- $\hat{C}_1 = \hat{A} + \hat{B}$ (Ext ∠ of $\Delta$)
If $AB = AC = BC$, then $\hat{A} = \hat{B} = \hat{C} = 60^\circ$

$\triangle ABC$ is **equilateral**

5. **When you must prove two sides are equal**

To prove $AC = CB$, prove $\hat{C}, = 90^\circ$

To prove $AB = AC$, prove $\hat{B} = \hat{C}$

To prove $AD = BD$, try prove $\triangle ACD \equiv \triangle BCD$

6. **Centre of a circle given**

7. **Diameter given**

If $AOB$ is diameter then $\hat{C} = 90^\circ$

($\angle$ in semi circle)
8. **Angles formed at the circumference**

If A, B, C and D are concyclic (lie on a circle) and if AB, AC, BD and CD are chords of the circle, then \( \hat{A} = \hat{D} \) and \( \hat{B} = \hat{C} \)

(\( \angle \) in same segment) or

(line/arc subtends equal angles)

9. **Chords in a circle**

If AB = BC, then \( \hat{D}_1 = \hat{D}_2 \)

If ABC and DEF are equal circles, then \( \hat{A} = \hat{F} \) if BC = DE

10. **Cyclic quadrilateral given**

If ABCD is cyclic then \( \hat{A} + \hat{C} = 180^\circ \) and

\( \hat{B} + \hat{D} = 180^\circ \)

(opposite \( \angle \)s of cyclic quad)

If LMOP is cyclic then \( \hat{M}_2 = \hat{P} \)

(Exterior \( \angle \)s of cyclic quad)

If EFGH is cyclic then

\( \hat{E}_1 = \hat{H}_1, \ \hat{E}_2 = \hat{F}_2, \)

\( \hat{F}_2 = \hat{G}_2, \ \hat{G}_1 = \hat{H}_1 \)

(\( \angle \)s in same segment)
11. **How to prove that a quadrilateral is cyclic**

ABCD would be a cyclic quadrilateral if you could prove one of the following:

**Condition 1:**

\[
\left( \hat{A}_1 + \hat{A}_2 \right) + \left( \hat{C}_2 + \hat{C}_3 \right) = 180^\circ \text{ or } \\
\left( \hat{B}_1 + \hat{B}_2 \right) + \left( \hat{D}_1 + \hat{D}_2 \right) = 180^\circ
\]

**Condition 2:**

\[
\hat{C}_1 = \hat{A}_1 + \hat{A}_2
\]

**Condition 3:**

\[
\hat{A}_1 = \hat{B}_2 \text{ or } \hat{A}_2 = \hat{D}_1 \text{ or } \hat{B}_1 = \hat{C}_2 \text{ or } \\
\hat{D}_2 = \hat{C}_3
\]

12. **Tangents to circles given**

- **Tan⊥ rad**
- **Tangents from the same point**
- **Tan-chord (acute case)**
- **Tan-chord (obtuse case)**

13. **How to prove that a line is a tangent to a circle**

ABC is a tangent if \( \hat{OBC} = 90^\circ \)

DCE is a tangent if \( \hat{C}_1 = \hat{A} \)
REVISION EXERCISE

1. O is the centre of two concentric circles. AB is a chord of the larger circle and tangent to the smaller circle at M. OM is joined. If OA = 20 cm and OM = 12 cm, calculate the length of AB.

2. O is the centre of the circle. STU is a tangent at T. Chord BC = chord CT, $\hat{A} = 105^\circ$ and $\hat{CTU} = 40^\circ$. Calculate the size of:
   (a) $\hat{A}_2$
   (b) $\hat{A}_1$
   (c) $\hat{B}_1 + \hat{B}_2$
   (d) $\hat{C}_2$

3. ALB is a tangent to circle LMNP. AL//MP. Prove that:
   (a) LM = LP
   (b) LN bisects $\hat{MNP}$
   (c) LM is a tangent to circle MNQ

4. In the figure below, RQ and RB are tangents at the points Q and B respectively to the circle with centre O. The radius BO produced meets the circle at A and RQ produced at P. Let $\hat{Q}_1 = x$.
   (a) Prove that RBOQ is a cyclic quadrilateral.
   (b) Name, with reasons, FOUR other angles in the figure which are equal to $x$.
   (c) Express $\hat{P}$ in terms of $x$. 

ABC would be a tangent to the “imaginary” circle drawn through $E = \hat{E}$.
IMPORTANT CONCEPTS REQUIRED FOR TRIANGLE GEOMETRY

1. **Area of Triangles**

   (a) The height or altitude of a triangle is always relative to the chosen base.

   ![Diagram of triangle with height and base labeled](image)

   In all cases, the area of the triangles can be calculated by using the formula $\text{Area } \triangle ABC = \frac{1}{2} \text{(base)(height)}$.

   (b) Two triangles which share a common vertex have a common height.

   ![Diagram of two triangles sharing a vertex](image)

   $\text{Area } \triangle ABC = \frac{1}{2} \cdot BC \cdot h$
   $\text{Area } \triangle ACD = \frac{1}{2} \cdot CD \cdot h$
   $\text{Area } \triangle \Delta DEG = \frac{1}{2} \cdot DG \cdot h$
   $\text{Area } \triangle FGE = \frac{1}{2} \cdot GF \cdot h$

   (c) Triangles with equal or common bases lying between parallel lines have the same area.

   ![Diagram of triangles with equal bases](image)

   $\text{Area } \triangle ABC = \text{Area } \triangle DBC$
   $\text{Area } \triangle EFH = \text{Area } \triangle HGE$
2. **Ratios**

Consider the line segment AB. If AB = 21 cm and C divides AB in the ratio AC:CB = 4 : 3, it is possible to find the actual lengths of AC and CB.

![Diagram of line segment AB divided into 4 parts and 3 parts](image)

It is clear that AC doesn’t equal 4 cm and CB doesn’t equal 3 cm because $4 + 3 \neq 21$ cm. However, if we let each part equal $k$, it will be possible to find the length of AC and CB in centimetres.

![Diagram of line segment AB divided into 4 parts and 3 parts](image)

The length of AC is $(4k)$ cm and the length of CB is $(3k)$ cm.

$4k + 3k = 21$ cm

$7k = 21$ cm

$k = 3$ cm

Each part represents 3 cm.

$AC = 4(3 \text{ cm}) = 12 \text{ cm}$

and $CB = 3(3 \text{ cm}) = 9 \text{ cm}$

![Diagram of line segment AB divided into 12 cm and 9 cm](image)

Note:

$$\frac{AC}{CB} = \frac{12 \text{ cm}}{9 \text{ cm}} = \frac{4}{3}$$

$4:3$ is the ratio of AC:CB.
3. **Cross multiplication techniques**

Cross multiplication is useful when working with ratios.

Consider, for example, the ratios \( \frac{3}{2} = \frac{6}{4} \)

The following statements are true for the given ratio:

If \( \frac{3}{2} = \frac{6}{4} \) then:

(a) \( \frac{2}{3} = \frac{4}{6} \) (invert both left and right)

(b) \( \frac{3}{6} = \frac{2}{4} \) (interchange 6 and 2)

(c) \( \frac{4}{2} = \frac{6}{3} \) (interchange 3 and 4)

(d) \( 3 \times 4 = 6 \times 2 \) (multiply 3 by 4 and 6 by 2)

(e) \( \frac{3 \times 4}{2} = 6 \) (multiply 3 by 4 only)

(f) \( 3 = \frac{6 \times 2}{4} \) (multiply 6 by 2 only)

In general, for the ratios \( \frac{a}{b} = \frac{c}{d} \)

If \( \frac{a}{b} = \frac{c}{d} \) then:

(a) \( \frac{b}{a} = \frac{d}{c} \)

(b) \( \frac{a}{c} = \frac{b}{d} \)

(c) \( \frac{d}{b} = \frac{c}{a} \)

(d) \( ad = bc \)

(e) \( \frac{ad}{b} = c \)

(f) \( a = \frac{bc}{d} \)

Consider the lengths AB, DE, AC and DF:

If \( \frac{AB}{DE} = \frac{AC}{DF} \) then:

(a) \( \frac{DE}{AB} = \frac{DF}{AC} \)

(b) \( \frac{AB}{AC} = \frac{DE}{DF} \)

(c) \( \frac{DF}{DE} = \frac{AC}{AB} \)

(d) \( AB \times DF = AC \times DE \)

(e) \( \frac{AB \times DF}{DE} = AC \)

(f) \( AB = \frac{AC \times DE}{DF} \)
PROPORTIONALITY THEOREMS

INVESTIGATION 1

Measure the length of $AD$, $DB$, $AB$, $AE$, $EC$, $AC$ and the size of $\hat{D}$ and $\hat{B}$.

(a) What can you conclude about $DE$ and $BC$?

(b) Complete: \[ \frac{AD}{DB} = \frac{AE}{EC} = \]

(c) What can you conclude about these ratios?

(d) Complete: \[ \frac{AB}{AD} = \frac{AC}{AE} = \]

(e) What can you conclude about these ratios?

We can now prove this conjecture in all cases. Theorem 1 will now be discussed.
THEOREM 1

A line drawn parallel to one side of a triangle cuts the other two sides so as to divide them in the same proportion.

**Given:** \( DE \parallel BC \)

**Required to prove:** \( \frac{AD}{DB} = \frac{AE}{EC} \)

**Proof:**

In \( \triangle ADE \), draw height \( h \) relative to base \( AD \) and height \( k \) relative to base \( AE \). Join \( BE \) and \( DC \) to create \( \triangle BDE \) and \( \triangle CED \).

\[
\frac{\text{Area } \triangle ADE}{\text{Area } \triangle BDE} = \frac{AD}{BD} = \frac{AE}{k}
\]

\[
\frac{\text{Area } \triangle ADE}{\text{Area } \triangle CED} = \frac{AE}{EC} = \frac{AD}{BD}
\]

Now it is clear that

\[
\text{Area } \triangle BDE = \text{Area } \triangle CED
\]

(same base, height and lying between parallel lines)

\[
\therefore \frac{\text{Area } \triangle ADE}{\text{Area } \triangle BDE} = \frac{\text{Area } \triangle ADE}{\text{Area } \triangle CED}
\]

\[
\therefore \frac{AD}{BD} = \frac{AE}{EC}
\]

**COROLLARIES**

1. \( \frac{AB}{AD} = \frac{AC}{AE} \)
2. \( \frac{AB}{DB} = \frac{AC}{EC} \)
3. \( \frac{BD}{DA} = \frac{CE}{EA} \)
4. \( \frac{BD}{BA} = \frac{CE}{CA} \)

Whenever you use this theorem the reason you must give is:

**Line \parallel one side of \( \triangle \).**
EXAMPLE 1

In $\triangle ADG$, $AG \parallel BF \parallel CE$.

(a) In $\triangle BDF$: \[
\frac{BC}{CD} = \frac{FE}{ED}
\]

(b) In $\triangle BDF$: \[
\frac{BD}{CD} = \frac{FD}{ED}
\]

(c) In $\triangle ADG$: \[
AB:BD = GF:FD
\]

(d) In $\triangle ADG$: \[
AD:BD = GD:FD
\]

EXAMPLE 2

In $\triangle ABC$, DE $\parallel$ BC, $AB = 28 \text{ mm}$ and $AE : EC = 4 : 3$. Determine the length of $BD$.

Solution

\[
\begin{array}{c|c}
\text{Statement} & \text{Reason} \\
\hline
\frac{BD}{28 \text{ mm}} = \frac{3k}{7k} & \text{Line } \parallel \text{ one side of } \triangle ABC \\
\therefore BD = \frac{3}{7} \times 28 \text{ mm} & \\
\therefore BD = 12 \text{ mm} & \\
\end{array}
\]
EXAMPLE 3

D and E are points on sides AB and BC respectively of \( \triangle ABC \) such that \( AD : DB = 2 : 3 \) and \( BE = \frac{4}{3} EC \). If \( DK \parallel AE \) and \( AE \) and \( CD \) intersect at \( P \), find the ratio of \( CP : PD \).

Solution

\[
\begin{align*}
\frac{CP}{PD} &= \frac{3p}{8p} & \text{Line} \parallel \text{one side of } \triangle CDK \\
\frac{EK}{4p} &= \frac{AD}{AB} & \text{Line} \parallel \text{one side of } \triangle ABE \\
\therefore \frac{EK}{4p} &= \frac{2k}{5k} & \therefore \frac{CP}{PD} = \frac{3p}{8p} \\
\therefore EK &= \frac{2k}{5k} \times 4p & \therefore \frac{CP}{PD} = \frac{3p \times 5}{8p} \\
\therefore EK &= \frac{2}{5} \times 4p & \therefore \frac{CP}{PD} = \frac{15}{8} \\
\therefore EK &= \frac{8p}{5} & \therefore \frac{CP}{PD} = \frac{15}{8}
\end{align*}
\]
EXAMPLE 4

In \( \triangle ABC \), \( AB \parallel FD \), \( AF \parallel DE \) and \( FE : EC = 3 : 4 \).
Determine the ratio \( EC : BC \).

**Solution**

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{EC}{BC} = \frac{4k}{BC} )</td>
<td>since ( EC = 4k )</td>
</tr>
<tr>
<td>Now ( \frac{BC}{FC} = \frac{AC}{DC} )</td>
<td>Line ( \parallel ) one side of triangle; ( DF \parallel AB )</td>
</tr>
<tr>
<td>Also ( \frac{AC}{DC} = \frac{7}{4} )</td>
<td>Line ( \parallel ) one side of triangle; ( AF \parallel DE )</td>
</tr>
<tr>
<td>( \frac{BC}{FC} = \frac{7}{4} )</td>
<td></td>
</tr>
<tr>
<td>( \therefore BC = \frac{7}{4} \times FC )</td>
<td></td>
</tr>
<tr>
<td>( \therefore BC = \frac{7}{4} \times 7k )</td>
<td></td>
</tr>
<tr>
<td>( \therefore BC = \frac{49k}{4} )</td>
<td></td>
</tr>
<tr>
<td>( \therefore \frac{EC}{BC} = \frac{4k}{BC} = \frac{4k}{\frac{49k}{4}} = \frac{4k \times 4}{49k} = \frac{16}{49} )</td>
<td></td>
</tr>
<tr>
<td>( \therefore \frac{EC}{BC} = \frac{16}{49} )</td>
<td></td>
</tr>
</tbody>
</table>
EXERCISE 1

1. In ∆PQR, ST || QR.
   Calculate the value of y.

2. In ∆ACE, BD || AE. Calculate the value of y.

3. In ∆DEF, GH || EF.
   Calculate the value of m.

4. In ∆ACF, AF || BG and in ∆CEF, EF || DG.
   ED = 22 cm, DC = 33 cm, BC = 15 cm and AB = x.
   Calculate the value of x.

   DE = 32 mm and DF = 24 mm.
   Find the length of DG, GE, DH and HF.

6. In ∆PQR, ST || QR. PQ = 35 mm, PR = 25 mm and QS = 14 mm.
   Determine the length of PT.
7. In $\triangle ABC$, $AC \parallel DE$. In $\triangle BCG$, $CG \parallel EF$. Prove that:
$AD:DB = GF:FB$.

8. In $\triangle ABC$, $DE \parallel AB$ and $DF \parallel AC$. Prove that $GB:GD = GD:GC$.

9. $DB \parallel FC$, $CE \parallel FD$ and $AG = \frac{1}{2} GF$.
The diagonals of parallelogram $GCFD$ intersect at $H$. Calculate:
1. $GH:HF$
2. $AG:GH$
3. $AB:BC$
4. $AE:ED$

10. In $\triangle ACE$, $BF \parallel CE$, $BC = \frac{3}{8} AC$ and $AE:ED = 4:3$.
Determine $DG:GB$.

11. In $\triangle PQR$, $PS \parallel VT$ and $QS:SR = 2:3$. $T$ is a point on $PR$ such that $PT:TR = 2:7$.
Prove that $QU:UT = 3:1$.

12. In $\triangle QRS$, $T$ and $U$ are points on $RS$ and $W$ is a point on $QS$ such that $QT \parallel UW$ and $QR \parallel WT$. If $QW = 12 \text{ mm}$, $WS = 11 \text{ mm}$ and $TU = 6 \text{ mm}$, find $RS$ correct to the nearest whole number.
13. In \( \triangle ABC \), \( XY \parallel AC \) and \( MN \parallel BC \). \( AN : NC = 3 : 2 \) and \( BY = 2YC \). \( AB = 15 \text{ cm} \). Calculate:
   (1) \( AM \)  
   (2) \( XB \)

14. \( \frac{RB}{RQ} = \frac{1}{3}, \) \( PA : AR = 1:2 \) and \( PM \parallel AB \).
   (1) Write down values for \( RA : RP \) and \( RB : BQ \)
   (2) Determine \( BM : BR \)
   (3) Prove that \( RM = MQ \)

15. \( Q \) bisects \( YZ \). \( SR \parallel XZ \).
   (1) Show that \( QR = RZ \)
   (2) Prove that \( YS = \frac{3}{4} \) \( YP \)
   (3) If \( YS = 12 \text{ cm} \), find the length of \( SP \).

16. \( PA = \frac{4}{9} \) \( PQ \) and \( 2PB = BR \).
   \( BC \parallel RA \). Determine:
   (1) \( BD : DQ \)
   (2) \( \frac{\text{Area } \triangle \text{PRA}}{\text{Area } \triangle \text{QRA}} \)
      (Hint: construct the common height)

17. In \( \triangle ABC \), \( P \) is the midpoint of \( AC \). \( RS \parallel BP \) and \( AR : AB = 3:5 \). Determine:
   (1) \( AS : SP \)
   (2) \( AS : SC \)
   (3) \( RT : TC \)
   (4) \( \frac{\text{Area } \triangle \text{TPC}}{\text{Area } \triangle \text{RSC}} \)
      (Hint: Use the area rule)
THEOREM 1 CONVERSE (not for examination purposes)

If a line cuts two sides of a triangle proportionally, then that line is parallel to the third side. Whenever you use this theorem the reason you must give is:

Line divides sides of Δ proportionally.

Given: \[ \frac{AD}{DB} = \frac{AE}{EC} \]

Required to prove: \( DE \parallel BC \)

Proof:

In \( \triangle ADE \), draw height \( h \) relative to base \( AD \) and height \( k \) relative to base \( AE \).

Join \( BE \) and \( DE \) to create \( \triangle BDE \) and \( \triangle CED \).

\[
\frac{\text{Area } \triangle ADE}{\text{Area } \triangle BDE} = \frac{\frac{1}{2} \cdot AD \cdot h}{\frac{1}{2} \cdot BD \cdot h} = \frac{AD}{BD}
\]

\[
\frac{\text{Area } \triangle ADE}{\text{Area } \triangle CED} = \frac{\frac{1}{2} \cdot AE \cdot k}{\frac{1}{2} \cdot EC \cdot k} = \frac{AE}{EC}
\]

But \( \frac{AD}{DB} = \frac{AE}{EC} \)

\[
\therefore \frac{\text{Area } \triangle ADE}{\text{Area } \triangle BDE} = \frac{\text{Area } \triangle ADE}{\text{Area } \triangle CED}
\]

\[
\therefore \text{Area } \triangle BDE = \text{Area } \triangle CED
\]

\[
\therefore DE \parallel BC \quad \text{(same base, same area)}
\]
EXAMPLE 5

In \( \triangle ABC \), \( AD = 10 \text{ cm} \), \( DB = 3 \text{ cm} \), \( AE = 15 \text{ cm} \) and \( EC = 4.5 \text{ cm} \).
Prove that \( DE \parallel BC \).

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{AD}{DB} = \frac{10}{3} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{AE}{EC} = \frac{15}{4.5} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{AD}{DB} = \frac{AE}{EC} )</td>
<td>Line divides sides of ( \triangle ABC ) prop</td>
</tr>
</tbody>
</table>


EXAMPLE 6

In \( \triangle ABC \) and \( \triangle ACD \), \( XY \parallel BC \) and \( YZ \parallel CD \). Prove that \( XZ \parallel BD \).

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>In ( \triangle ABC ):</td>
<td>Line ( \parallel ) one side of ( \triangle )</td>
</tr>
<tr>
<td>( \frac{AX}{XB} = \frac{AY}{YC} )</td>
<td></td>
</tr>
<tr>
<td>In ( \triangle ACD ):</td>
<td>Line ( \parallel ) one side of ( \triangle )</td>
</tr>
<tr>
<td>( \frac{AZ}{ZD} = \frac{AY}{YC} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{AX}{XB} = \frac{AZ}{ZD} )</td>
<td>Line divides sides of ( \triangle ABD ) prop</td>
</tr>
<tr>
<td>( \therefore XZ \parallel BD )</td>
<td></td>
</tr>
</tbody>
</table>
EXERCISE 2

1. Using information given on the diagram, prove that AB∥EF.

2. In the diagram below, KM is a diameter of the circle centre O. OK = r, OC = 4r and \(\hat{H} = \hat{C}\). Prove that EK∥HC.

3. O is any point inside \(\triangle PQR\). AB∥PQ and AC∥PR. Prove that BC∥QR.

4. P is any point on side BC of quadrilateral ABCD. A straight line through P, parallel to AB intersects AC at Q. A straight line through P, parallel to BD intersects DC at R. Prove that QR∥AD.

THEOREM 2 (THE MIDPOINT THEOREM)

This theorem is a special case of Theorem 1. It states that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and equal to half the third side. If \(AD = DB\) and \(AE = EC\), then \(DE∥BC\) and \(BC = 2DE\) or \(DE = \frac{1}{2}BC\).
THEOREM 2 CONVERSE (not for examination purposes)

The line passing through the midpoint of one side of a triangle, parallel to another side, bisects the third side and is equal to half the length of the side it is parallel to.
If \( AD = DB \) and \( DE \parallel BC \), then \( AE = EC \) and \( BC = 2DE \) or \( DE = \frac{1}{2} BC \).

EXAMPLE 7

In \( \triangle ABC \), \( AD = DB \) and \( AE = EC \). DE is produced to \( F \). \( DB \parallel FC \). BC = 32 mm.
(a) Prove that \( DBCF \) is a parallelogram.
(b) Calculate the length of \( DE \).

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) ( DB \parallel FC ) ( DE \parallel BC )</td>
<td>Given</td>
</tr>
<tr>
<td>( \therefore DF \parallel BC )</td>
<td>Since in ( \triangle ABC ): ( AD = DB ) and ( AE = EC ), Midpoint theorem</td>
</tr>
<tr>
<td>( \therefore DBCF ) is a parallelogram</td>
<td>DEF is a straight line, Opp sides parallel</td>
</tr>
<tr>
<td>(b) ( BC = 32 ) mm ( \therefore DE = 16 ) mm</td>
<td>Given, Midpoint theorem</td>
</tr>
</tbody>
</table>

EXERCISE 3

1. PQRS is a kite. A and B are the midpoints of PQ and PS respectively. \( QD = DR \) and \( SC = CR \). Prove that \( ABCD \) is a parallelogram.

2. In \( \triangle ACD \), \( AB = BC \), \( GE = 15 \) cm and \( AF = FE = ED \). Find \( CG \).
3. Determine the perimeter of $\triangle PQR$ in terms of $x$.

4. $\square PQRS$ is a trapezium, $PS \parallel AB \parallel QR$ and $SB = BR$. Draw $PR$ and then prove:
   (1) $PA = AQ$
   (2) $2AB = QR + PS$

5. In $\triangle ABC$, $PQ \parallel MN \parallel BC$.
   $CA = 2NA = 4QA$. Let $QA = x$.
   (a) Prove that $AN = NC$.
   (b) Calculate $PQ$ if $BC = 32$.

6. $\triangle PQT$ is inscribed in a circle.
   $AO \parallel QR$. $PA = AQ$ and $PB = BT$. Prove that:
   (a) $AB \parallel QT$
   (b) $O$ is the centre of the circle if $PR$ is a diameter.
   (c) $BORT$ is a trapezium.

7. $O$ is the centre of the circle. $D$ is any point on the circumference. $AD$ is produced its own length to $B$. Prove that $\hat{CAB} = \hat{CBA}$.
   (Let the radius equal $x$).
8. In \( \triangle PQR \), \( A \) is a point on \( PQ \) such that \( PA : AQ = 5 : 4 \). \( B \) is a point on \( PR \) such that \( PB : BR = 5 : 2 \). \( AB \) produced cuts \( QR \) produced in \( T \) and \( S \) is the midpoint of \( AQ \).
(a) Prove that \( AT = 2SR \)
(b) If \( RK \parallel QX \), determine \( PX : XT \)

**SIMILARITY THEOREMS**

**Revision of the similarity of polygons**

Two polygons are similar if they have the same shape but not necessarily the same size. For example, polygon \( EFGH \) is an enlargement of polygon \( ABCD \) by a scale factor of 2 and the two polygons are similar.

Two conditions must both be satisfied for two polygons to be similar:
(a) The corresponding angles must be equal.
(b) The ratio of the corresponding sides must be in the same proportion.

Polygon \( ABCD \) and \( EFGH \) are similar since both conditions hold true:
(a) \( \hat{A} = \hat{E} = 35^\circ \quad \hat{B} = \hat{F} = 155^\circ \quad \hat{C} = \hat{G} = 96^\circ \quad \hat{D} = \hat{H} = 76^\circ \)
(b) \( \frac{AB}{EF} = \frac{1.95}{3.9} = \frac{1}{2} \quad \frac{BC}{FG} = \frac{1}{2} \quad \frac{CD}{GH} = \frac{2}{4} = \frac{1}{2} \quad \frac{AD}{EH} = \frac{2.75}{5.5} = \frac{1}{2} \)

It is important to note that two polygons may well have their corresponding angles equal, but the corresponding sides may not necessarily be in the same proportion. For example, consider rectangle \( ABCD \) and square \( EFGH \). The corresponding angles are equal, but the corresponding sides are not in the same proportion. This is why it is necessary for both conditions to hold true.

With similar triangles, only one of the two conditions needs to be true in order for the two triangles to be similar. This will be proved in Theorem 3 which states that if the corresponding angles of two triangles are equal (triangles are equiangular)
then the corresponding sides will always be in the same proportion and the two triangles will be similar. This is unique to triangles only. Conversely, if the corresponding sides are in the same proportion, then the corresponding angles of the two triangles will be equal (Theorem 3 converse).

**THEOREM 3**

The corresponding sides of two equiangular triangles are in the same proportion and therefore the triangles are similar.

**Given:**

\[ \hat{A} = \hat{D}, \hat{B} = \hat{E} \text{ and } \hat{C} = \hat{F} \]

**Required to prove:**

\[
\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}
\]

**Proof:**

On AB mark off \( AG = DE \).
On AC mark off \( AH = DF \).
Join GH.
In \( \triangle AGH \) and \( \triangle DEF \):
(1) \( AG = DE \) construction
(2) \( \hat{A} = \hat{D} \) given
(3) \( AH = DF \) construction
\[
\therefore \triangle AGH \cong \triangle DEF \text{ SAS}
\]
\[
\therefore \hat{G} = \hat{E}
\]
But \( \hat{B} = \hat{E} \) given
\[
\therefore \hat{G} = \hat{B}
\]
\[
\therefore GH \parallel BC \text{ corresponding angles equal.}
\]
\[
\therefore \frac{AB}{AG} = \frac{AC}{AH}
\]
\[
\therefore \frac{AB}{DE} = \frac{AC}{DF} \quad (AG = DE, AH = DF)
\]
Similarly, by constructing BG and BH on AB and BC respectively, it can be proved that
\[
\frac{AB}{DE} = \frac{BC}{EF}
\]
\[
\therefore \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}
\]
Therefore the triangles are similar.
EXAMPLE 8

In \( \triangle PST \), \( TS \perp PS \) and \( RQ \perp PT \). Prove:

(a) \( \triangle PRQ \parallel \triangle PST \)
(b) \( RQ : PQ = ST : PT \)
(c) \( PR \cdot PT = PQ \cdot PS \)

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Match the corresponding angles of ( \triangle PRQ ) and ( \triangle PST ) as follows and then prove the pairs of angles equal.</td>
<td></td>
</tr>
<tr>
<td>( \hat{P} ) ( \parallel \hat{P} )</td>
<td>Draw solid lines for each pair of corresponding angles that are equal.</td>
</tr>
<tr>
<td>( \hat{R}_1 \parallel \hat{S}_1 + \hat{S}_2 )</td>
<td>The dotted line indicates that the pair of angles are equal due to the sum of the angles of a triangle.</td>
</tr>
<tr>
<td>( \hat{Q}_2 \parallel \hat{T} )</td>
<td></td>
</tr>
</tbody>
</table>

In \( \triangle PRQ \) and \( \triangle PST \):

(1) \( \hat{P} = \hat{P} \) | common |
(2) \( \hat{R}_1 = \hat{S}_1 + \hat{S}_2 = 90^\circ \) | given |
(3) \( \hat{Q}_2 = \hat{T} \) | sum of angles of \( \triangle \) |
\( \therefore \triangle PRQ \parallel \triangle PST \) | \( \angle, \angle, \angle \) |

(b) Since \( \triangle PRQ \parallel \triangle PST \):

\[
\begin{align*}
\frac{PR}{PS} &= \frac{RQ}{ST} &= \frac{PQ}{PT} \\
\therefore \frac{RQ}{ST} &= \frac{PQ}{PT} \\
\therefore RQ : PQ &= ST : PT \\
\end{align*}
\]

(c) \( \frac{PR}{PS} = \frac{PQ}{PT} \) since \( \frac{PR}{PS} = \frac{RQ}{ST} = \frac{PQ}{PT} \)

\( \therefore PR \cdot PT = PQ \cdot PS \) cross multiplication
EXAMPLE 9

A, B, C and D are concyclic points. DOC and AOB are chords. DB and AC are joined. Prove that:

(a) \( \triangle AOC \parallel \triangle DOB \)

(b) \( \frac{OB}{OD} = \frac{OC}{OA} \)

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
</table>
| (a) Match the corresponding angles of \( \triangle AOC \) and \( \triangle DOB \) as follows and then prove the pairs of angles equal. | arc BC subtends equal angles, arc AD subtends equal angles, sum of angles of \( \triangle \)
| \( \hat{A} \longrightarrow \hat{D} \) | \( \angle, \angle, \angle \) |
| \( \hat{O}_2 \cdots \cdots \hat{O}_1 \) | corr sides of \( \triangle \)'s in proportion |
| \( \hat{C} \longrightarrow \hat{B} \) | cross multiplication |

Note: You could have also used the reason “vertically opposite angles” for statement (3) above.
EXERCISE 4

1. In the diagram, PA || BC and \( \hat{B}_1 = \hat{C} \). Prove that:
   (a) \( \triangle PAB \parallel \triangle ABC \)
   (b) \( PA : AB = PB : AC \)
   (c) \( AB : AC = BP : BC \)

2. In the diagram, ABCD is a parallelogram. \( BE = BC \)
   and \( BD = AB \). \( \hat{A} = x \).
   Prove that:
   (a) \( \triangle ABD \parallel \triangle CBE \)
   (b) \( AB \cdot BE = BD \cdot BC \)
   (c) \( \frac{BD}{AD} = \frac{BE}{CE} \)

3. In the diagram, EF || CGD and \( BEC \parallel FG \). Prove that:
   (a) \( \triangle BEF \parallel \triangle FGD \)
   (b) \( \frac{DG}{EF} = \frac{FG}{BE} \)

4. PB is a tangent to circle ABC. PA || BC. Prove that:
   (a) \( \triangle PAB \parallel \triangle ABC \)
   (b) \( PA : AB = AB : BC \)
   (c) \( \frac{AP}{BP} = \frac{AB}{AC} \)

5. AB is a diameter of circle ABC. DC is a tangent at C and \( BD \perp CD \).
   (a) Prove that \( \triangle BDC \parallel \triangle BCA \)
   (b) If \( AB = 25 \text{cm} \) and \( AC = 24 \text{cm} \)
   calculate the length of:
   (1) \( BC \) (2) \( CD \)

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In the examples which follow, you are required to locate triangles. The following strategy will help you in these types of examples.

1. Try taking letters from the top and bottom and see if you can locate similar triangles.

\[
\begin{align*}
\frac{BC}{AC} &= \frac{AB}{AD}
\end{align*}
\]

Refer to the diagram and see which triangles appear:
Is ABC a triangle?
Is ABD a triangle?

If there are two triangles, match the angles:

\[
\begin{align*}
\hat{A}_1 &\sim \hat{A}_2 \\
\hat{B}_1 &\sim \hat{B}_2 \\
\hat{C} &\sim \hat{D}
\end{align*}
\]

When proving that the two triangles are similar, make sure that the equal angles correspond:
\(\triangle ABC \parallel \triangle DBA\)

Use the theorems proved in previous grades to prove that two pairs of corresponding angles are equal. The third pair will always be equal due to the sum of the angles of a triangle.

2. Try taking letters from the left and right and see if you can locate similar triangles.

\[
\begin{align*}
\frac{BC}{AC} &= \frac{AB}{AD}
\end{align*}
\]

Refer to the diagram and see which triangles appear:
Is ABC a triangle?
Is ACD a triangle?

3. If you cannot locate triangles or if one triangle is acute-angled and the other one contains an obtuse angle making it impossible for the triangles to be similar, try one or more of the following strategies:
(a) Replace lengths with equal other lengths and then try to locate triangles.
(b) Use information from previous parts of the question to assist you.
(c) Look for other pairs of triangles which might be similar and have a bearing on what you are trying to prove.
EXAMPLE 10

Circles ABC and ABD intersect at A and B respectively. AC is a tangent to circle ABD. AD is a tangent to circle ABC. Prove that:

(a)  \[ \frac{BC}{AC} = \frac{AB}{AD} \]

(b)  \[ AB^2 = BC \cdot BD \]

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
</table>
| (a) \[
\frac{BC}{AC} = \frac{AB}{AD}
\] (Try letters from left to right) | |
| ABC is a triangle (see diagram) | |
| ACD is a not a triangle (see diagram) | |
| Now write the angles of each triangle as follows: | |
| \( \hat{A}_1 = \hat{A}_2 \) | Draw solid lines for each pair of corresponding angles that are equal. |
| \( \hat{B}_1 = \hat{B}_2 \) | The dotted line indicates that the pair of angles are equal due to the sum of the angles of a triangle. |
| \( \hat{C} = \hat{A}_2 \) | |
| Now prove that the triangles are similar. Make sure the angles that are equal correspond. | |
| \[ \Delta ABC \parallel \Delta DBA \] | |
| (1) \( \hat{A}_1 = \hat{D} \) | tan-chord |
| (2) \( \hat{C} = \hat{A}_2 \) | tan-chord |
| (3) \( \hat{B}_1 = \hat{B}_2 \) | sum of angles of \( \Delta \) |
| \( \angle, \angle, \angle \) | |
| \[ \Delta ABC \parallel \Delta DBA \] | |
| \[ \frac{AB}{DB} = \frac{BC}{BA} = \frac{AC}{DA} \] | corr sides of \( \Delta 's \) in proportion |
EXAMPLE 11

PQ and PS are tangents to circle QRST. ST, RS and RQ are chords and QS is joined. PQ\parallel ST. Prove that: $QR^2 = RS \cdot RP$

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\therefore \frac{BC}{BA} = \frac{AC}{DA}$</td>
<td>Cross multiplication</td>
</tr>
<tr>
<td>$\therefore \frac{BC}{BA} = \frac{AC}{DA}$</td>
<td>DA = AD and BA = AB</td>
</tr>
<tr>
<td>$\therefore \frac{AB}{DA} = \frac{AC}{AD}$</td>
<td>corr sides of Δ's in proportion</td>
</tr>
<tr>
<td>$\therefore \frac{AB}{DB} = \frac{BC}{BA}$</td>
<td>Cross multiplication</td>
</tr>
<tr>
<td>$\therefore AB \times BA = BC \times DB$</td>
<td>AB = BA and DB = BD</td>
</tr>
<tr>
<td>$\therefore AB^2 = BC \cdot BD$</td>
<td></td>
</tr>
</tbody>
</table>

You are required to prove that $QR^2 = RS \cdot RP$
This can be written as $QR \cdot QR = RS \cdot RP$ or alternatively as follows:

Let’s locate triangles:

- $\hat{Q}_2 \Rightarrow \hat{P}_1$
- $\hat{R}_2 + \hat{R}_3 \Rightarrow \hat{Q}_1$
- $\hat{S}_2 \Rightarrow \hat{R}_1$

In ΔQRS\parallel ΔPRQ

(a) $\hat{Q}_2 = \hat{T}$
$\hat{T} = \hat{P}_1$
$\therefore \hat{Q}_2 = \hat{P}_1$
EXERCISE 5

1. ABCD is a cyclic quadrilateral. AB and DC produced meet at E. Prove that: \[
\frac{AE}{CE} = \frac{AD}{BC}
\]

2. AD is a diameter of circle ABDC. AE \perp BC. AB, AD, AC and DC are chords. Prove that: \[
\frac{AB}{BE} = \frac{AD}{DC}
\]

3. PT is a tangent to circle BAT. BA is produced to P. TB is joined. AT is a chord. Prove that: \[
\frac{BT}{BP} = \frac{AT}{PT}
\]
4. AOB is a diameter of circle centre O. Chord AC produced meets tangent BD at D. BC is joined. Prove that:
   \[ AB^2 = AC \cdot AD \]

5. ABCD is a cyclic quadrilateral, AC and BD intersect at P. E is a point on BD such that AE \parallel DC. Prove that:
   (a) \[ \frac{AP}{PC} = \frac{PE}{PD} \]
   (b) \[ AP^2 = BP \cdot PE \]

6. ABCD is a trapezium. Prove that:
   (a) \[ \frac{AD}{DE} = \frac{DE}{BE} \]
   (b) \[ DE \cdot CE = AE \cdot BE \]

7. The bisector of \( \hat{K} \) in \( \triangle KLM \) cuts LM at H and meets the circle circumscribing \( \triangle KLM \) at T. Prove that:
   (a) \[ TL^2 = TK \cdot TH \]
   (b) \[ KH \cdot LT = KM \cdot LH \]

8. M is the centre of circle ABCD. AB, AD, BD and DC are chords of the circle. MC bisects BMD and \( \hat{B}_1 = \hat{D}_1 \). Prove that:
   (a) \[ BD \cdot MC = AD \cdot DC \]
   (b) \[ \frac{AB}{DM} = \frac{BD}{CD} \]
MR is a tangent to circle MST. Chord ST is produced to R.

(a) Prove that $RM^2 = RT \cdot RS$

(b) If $RS = x$, calculate the length of RS if $TS = 5$ units and $RM = 6$ units

**EXAMPLE 12**

ABCD is a cyclic quadrilateral. Chord BC = Chord DC. ECF is a tangent to the circle at C. Chord AB is produced to meet the tangent at E. AC and BD are drawn. Prove that:

(a) $BD \parallel EF$

(b) \[
\frac{CD}{BE} = \frac{AD}{CD}
\]

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
</table>
| (a) $\hat{B}_2 = \hat{D}_2$
$\hat{D}_2 = \hat{C}_1$
$\therefore \hat{B}_2 = \hat{C}_1$
$\therefore BD \parallel EF$ | angles opp equal sides
| $\frac{CD}{BE} = \frac{AD}{CD}$ | tan-chord
| $ACD$ is a triangle
$BEC$ is not a triangle
BECD is not a triangle
ACD is a triangle | alt angles equal

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<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>It is clear that we don’t have two triangles that can be proved similar. Let’s try to replace a length with an equal length. In this example, BC = CD</td>
<td></td>
</tr>
<tr>
<td>[ \frac{CD}{BE} = \frac{AD}{CD} ] [ \frac{BC}{BE} = \frac{AD}{CD} ]</td>
<td></td>
</tr>
<tr>
<td>Now let’s try to locate triangles.</td>
<td></td>
</tr>
<tr>
<td>BC \quad BE \quad AD \quad CD</td>
<td></td>
</tr>
<tr>
<td>BCE is a triangle</td>
<td></td>
</tr>
<tr>
<td>ACD is a triangle</td>
<td></td>
</tr>
<tr>
<td>In ( \triangle BCE \parallel \triangle DAC )</td>
<td></td>
</tr>
<tr>
<td>(1) ( \hat{B}_1 = \hat{D}_1 + \hat{D}_2 )</td>
<td>ext angle of cyclic quad</td>
</tr>
<tr>
<td>(2) ( \hat{C}_1 = \hat{B}_2 )</td>
<td>alt angles equal ; BD \parallel EF</td>
</tr>
<tr>
<td>( \hat{B}_2 = \hat{A}_2 )</td>
<td>arc CD subtends equal angles</td>
</tr>
<tr>
<td>( \therefore \hat{C}_1 = \hat{A}_2 )</td>
<td></td>
</tr>
<tr>
<td>(3) ( \hat{E} = \hat{C}_3 )</td>
<td>sum of the angles of triangle</td>
</tr>
<tr>
<td>( \therefore \triangle BCE \parallel \triangle DAC )</td>
<td>( \angle, \angle, \angle )</td>
</tr>
<tr>
<td>( \frac{BC}{DA} = \frac{CE}{AC} = \frac{BE}{DC} )</td>
<td>corr sides of ( \triangle )'s in proportion</td>
</tr>
<tr>
<td>( \therefore \frac{BC}{DA} = \frac{BE}{DC} )</td>
<td></td>
</tr>
<tr>
<td>( \therefore \frac{BC}{AD} = \frac{CD}{BE} )</td>
<td>DA = AD and DC = CD</td>
</tr>
<tr>
<td>( \therefore \frac{CD}{BE} = \frac{AD}{CD} )</td>
<td>cross multiplication</td>
</tr>
<tr>
<td>( \therefore \frac{BC}{BE} = \frac{CD}{AD} )</td>
<td></td>
</tr>
<tr>
<td>( \therefore \frac{CD}{BE} = \frac{AD}{CD} )</td>
<td>BC = CD</td>
</tr>
</tbody>
</table>
EXAMPLE 13

In the diagram, CD ⊥ AB and AF ⊥ BC.

(a) Prove that \( \frac{CE}{CF} = \frac{CB}{CD} \)

(b) Prove that \( AE^2 = CE \cdot DE \) if it is given that \( AE = FE \).

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
</table>
| (a) In \( \triangle CEF \) and \( \triangle CBD \):  
1. \( \hat{C} = \hat{C} \)  
2. \( \hat{F}_2 = 90^\circ \)  
3. \( \hat{D}_2 = 90^\circ \)  
   \[ \therefore \hat{F}_2 = \hat{D}_2 \]  
   \[ \therefore \hat{E}_1 = \hat{B} \]  
   \[ \therefore \triangle CEF \parallel \triangle CBD \]  
   \[ \therefore \frac{CE}{EF} = \frac{CF}{BD} \]  
   \[ \therefore \frac{CE}{CB} = \frac{CF}{CD} \] | common  
adj suppl angles  
adj suppl angles  
sum of angles of \( \triangle \)  
\( \angle, \angle, \angle \)  
corr sides of \( \triangle \)'s in proportion  
cross multiplication |
| (b) You are required to prove that \( AE^2 = CE \cdot DE \)  
This can be written as \( AE \cdot AE = CE \cdot DE \)  
or alternatively as follows: \( \frac{AE}{DE} = \frac{CE}{AE} \)  
Let's locate triangles: \( \frac{AE}{DE} = \frac{CE}{AE} \)  
ADE is a triangle  
ACE is not a triangle  
ACE is not a triangle  
ADE is a triangle  
Let's now replace a length with an equal length and then try to locate triangles. |
EXERCISE 6

1. Rectangle DEFK is drawn inside right angled ΔABC such that D and E are on the sides of ΔABC. KF is on side BC. Prove that:
   (a) $\frac{AD}{BD} = \frac{DE}{BK}$
   (b) $\frac{DE}{EC} = \frac{AE}{FC}$
   (c) $\frac{KF}{EC} = \frac{AE}{FC}$
   (d) $\frac{AB}{AC} = \frac{AE}{DE}$

2. In trapezium ABCD, $DC = 2BC$, $\hat{A} = \hat{E_1}$ and $BC = EC$. Prove that:
   (a) $\frac{AD}{BD} = \frac{EC}{DC}$
   (b) $BD = 2AD$
3. TSQR is a cyclic quadrilateral. 
SR || PQ and TQ bisects \( \hat{PTR} \).
Prove that:
(a) PQ is a tangent to the circle.
(b) SQ = QR
(c) \( QS^2 = TR \cdot SP \)

4. Circle ABD and ACE intersect at A, C and D. AE and AB are tangents.
Prove that:
(a) AC = AD
(b) \( AD^2 = BC \cdot DE \)

5. AB is a tangent to the circle BCFGD at B and GF || EA. Chord DC is produced to A.
Prove that:
(a) \( AB^2 = AD \cdot AC \)
(b) \( AE^2 = AC \cdot AD \)
(c) \( AB = AE \)
(d) \( \frac{CE}{ED} = \frac{AB}{AD} \)

6. FA and FB are tangents to circle ABC with \( BC = AC \cdot FD \parallel CB \)
and \( \hat{CAF} = x \). Chord AB is produced to D and chord AC is produced to meet DF at E. BC is joined.
(a) Write down five other angles equal to x.
(b) Hence deduce that:
   (1) ABEF is a cyclic quadrilateral
   (2) AF = BD
(c) Prove that \( AF : FB = DE : AE \)
THEOREM 3 CONVERSE (not for examination purposes)

If the corresponding sides of two triangles are in the same proportion, the two triangles are equiangular and therefore similar.

\[ \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} \]

**Given:**

**Required to prove:**

\[ \Delta ABC \parallel \Delta DEF \]

**Proof:**

Construct \( \Delta ABC \parallel \Delta GEF \) (as in diagram)

\[ \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{GF} \]

But \[ \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} \]

\[ \therefore \frac{AB}{DE} = \frac{BC}{EF} \quad \text{(both equal \( \frac{BC}{EF} \))} \]

\[ \therefore GE = DE \]

Similarly, it can be proved that \( GF = DF \)

Therefore it can be concluded that \( \Delta DEF \equiv \Delta GEF \) (SSS).

\[ \therefore \Delta DEF \parallel \Delta GEF \]

But \( \Delta ABC \parallel \Delta GEF \)

\[ \therefore \Delta ABC \parallel \Delta DEF \]

**EXAMPLE 14**

In \( \Delta DGH \), \( EF = 5 \), \( DE = 4 \), \( EG = 8 \), \( DF = 6 \), \( FH = 2 \) and \( GH = 10 \).

Prove that \( \Delta DEF \parallel \Delta DHG \).

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{DE}{DH} = \frac{4}{8} = \frac{1}{2} )</td>
<td>Corr sides of triangles in same proportion</td>
</tr>
<tr>
<td>( \frac{EF}{EG} = \frac{5}{10} = \frac{1}{2} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{DF}{DG} = \frac{6}{12} = \frac{1}{2} )</td>
<td></td>
</tr>
<tr>
<td>( \therefore \frac{DE}{EF} = \frac{DF}{DG} )</td>
<td></td>
</tr>
<tr>
<td>( \therefore \Delta DEF \parallel \Delta DHG )</td>
<td></td>
</tr>
</tbody>
</table>
EXERCISE 7

1. In \( \triangle ABC \), \( AB = 4.8 \),
   \( AC = 3.6 \) and \( BC = 4.2 \).
In \( \triangle DEF \), \( DE = 3.2 \),
   \( DF = 2.4 \) and \( EF = 2.8 \).
Prove that \( \triangle ABC \parallel \triangle DEF \).

2. In \( \triangle TRP \), \( QR = 9.6 \), \( TP = 4.5 \),
   \( PS = 1.5 \), \( SR = 12 \), \( SQ = 4 \) and
   \( TS = 3.6 \). Prove that:
   (a) \( \triangle STP \parallel \triangle QRS \)
   (b) PT is a tangent to the circle
       TSR
   (c) QS \parallel TP
   (d) TQ = 1,2

3. ABCD is a quadrilateral.
   If \( AE \cdot BE = CE \cdot DE \) and
   \( DE \cdot CB = BE \cdot AD \),
prove that:
   (a) \( \triangle AED \parallel \triangle CEB \)
   (b) \( ED : EB = 1:2 \) if it is
given that \( AD = \frac{1}{2} CB \).

THEOREM 4

The perpendicular drawn from the vertex of the right angle of a right-angled
triangle to the hypotenuse, divides the triangle into two triangles that are similar
to each other and similar to the original triangle.

**Given:** \( \hat{A}_1 + \hat{A}_2 = 90^\circ \) and \( \hat{D}_2 = 90^\circ \)

**Required to prove:**

\( \triangle ABC \parallel \triangle DBA \parallel \triangle DAC \)
**Proof:**

In \( \triangle ABC \) and \( \triangle DBA \):

(a) \( \hat{A} = \hat{D}_1 = 90^\circ \) given
(b) \( \hat{B} = \hat{B} \) common
(c) \( \hat{C} = \hat{A}_1 \) sum of \( \angle \)'s \( \Delta \)

\( \therefore \triangle ABC \parallel \triangle DBA \)

\( \therefore \triangle ABC \parallel \triangle DAC \)

\[ \therefore \triangle ABC \parallel \triangle DBA \parallel \triangle DAC \]

**Corollaries**

<table>
<thead>
<tr>
<th>( \triangle ABC \parallel \triangle DBA )</th>
<th>( \triangle ABC \parallel \triangle DAC )</th>
<th>( \triangle DBA \parallel \triangle DAC )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \therefore \frac{AB}{DB} = \frac{BC}{BA} = \frac{AC}{DA} )</td>
<td>( \therefore \frac{AB}{DA} = \frac{BC}{AC} = \frac{AC}{DC} )</td>
<td>( \therefore \frac{DB}{DA} = \frac{BA}{AC} = \frac{DA}{DC} )</td>
</tr>
<tr>
<td>( \therefore AB^2 = BD \cdot BC )</td>
<td>( \therefore AC^2 = CD \cdot CB )</td>
<td>( \therefore AD^2 = BD \cdot DC )</td>
</tr>
</tbody>
</table>

**THEOREM 5** (The Theorem of Pythagoras)

By using the results of Theorem 4, prove that \( BC^2 = AB^2 + AC^2 \).

From the corollaries:

\( AB^2 = BD \cdot BC \) and \( AC^2 = CD \cdot CB \)

\( \therefore AB^2 + AC^2 = BD \cdot BC + CD \cdot CB \)

\( \therefore AB^2 + AC^2 = BC(BD + CD) \)

\( \therefore AB^2 + AC^2 = BC(BC) \)

\( \therefore AB^2 + AC^2 = BC^2 \)

\( \therefore BC^2 = AB^2 + AC^2 \)

**EXAMPLE 15**

In \( \triangle ABC \), \( \hat{C} = 90^\circ \) and \( CD \perp AB \) at \( D \).

DF \perp AC at \( F \) and \( DE \perp BC \) at \( E \).

(a) Why is \( \triangle DBE \parallel \triangle CDE \)?

(b) Complete the following for \( \triangle ABC \) :

\( \begin{align*}
(1) & \quad DB^2 = \ldots \times \ldots \\
(2) & \quad DC^2 = \ldots \times \ldots \\
(3) & \quad DE^2 = \ldots \times \ldots \\
\end{align*} \)

(c) Why is \( DC^2 = CF \cdot CA \) in \( \triangle ACD \)?

(d) Prove that \( \frac{BD}{AD} = \frac{BC^2}{AC^2} \).
<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
</table>
| (a) $\triangle DBE \parallel \triangle ACDE$                       | In $\triangle DBC$:
|                                 | $\hat{BDC} = 90^\circ$,  
|                                 | $\hat{DEB} = 90^\circ$  
|                                 | Perpendicular from right angled vertex to hypotenuse                   |
| (b) (1) $DB^2 = BE \times BC$                                       | In $\triangle ACD$:
| (2) $DC^2 = CE \times CB$                                           | $\hat{ADC} = 90^\circ$,  
| (3) $DE^2 = BE \times EC$                                           | $\hat{DFA} = 90^\circ$  
|                                 | Perpendicular from right angled vertex to hypotenuse                   |
| (c) $DC^2 = CF \times CA$                                            | In $\triangle ABC$:
|                                 | $\hat{ADC} = 90^\circ$,  
|                                 | $\hat{ACB} = 90^\circ$  
|                                 | Perpendicular from right angled vertex to hypotenuse                   |
| (d) $BC^2 = BD \times BA$                                            | $\triangle ABC$                                                       |
| $AC^2 = AD \times AB$                                                | $\triangle ABD$                                                       |
| $\therefore \frac{BC^2}{AC^2} = \frac{BD \times BA}{AD \times AB}$ | $\triangle ABD$                                                       |
| $\therefore \frac{BC^2}{AC^2} = \frac{BD \times AB}{AD \times AB}$  | $\triangle ABD$                                                       |
| $\therefore \frac{BC^2}{AC^2} = \frac{BD}{AD}$                     | $\triangle ABD$                                                       |

**EXERCISE 8**

1. O is the centre of the circle. AB is a diameter. Chord CP is produced to B. 
   Prove that: $AP^2 = PC \times BP$  

2. AD is a diameter of circle ABD. DC is a tangent and $EC \perp AC$. 
   Prove that:
   (a) $CD^2 = AD \times DE$  
   (b) $CD^2 = AC \times BC$
3. AB and AC are tangents to the circle centre O. OD \perp BC.
Prove that:
(a) \( BD^2 = OD \cdot DA \)
(b) \( \frac{OC^2}{AC^2} = \frac{OD}{DA} \)

4. ABOC is a kite in which \( \hat{B} = \hat{C} = 90^\circ \).
(a) Why is \( \triangle OCD \parallel \triangle OAC \)?
(b) Hence complete:
   (1) \( OC^2 = \ldots \times \ldots \)
   (2) \( CA^2 = \ldots \times \ldots \)
   (3) \( CD^2 = \ldots \times \ldots \)
(c) Prove that:
   (1) \( \frac{BD^2}{OB^2} = \frac{AD}{AO} \)
   (2) \( OC^2 - OD^2 = OD \cdot DA \)
(d) If \( OD = \frac{1}{2} DA = x \), prove that \( CD = \sqrt{2} \cdot OD \)

5. In \( \triangle ABC \), \( \hat{A} = 90^\circ \), \( AD \perp BC \),
   \( BD = 3 \), \( DC = x \) and \( AC = 2 \).
   Calculate the length of:
   (a) \( DC \)
   (b) \( AB \) (simplest surd form)

6. In \( \triangle ADC \), \( \hat{A\hat{D}C} = 90^\circ \) and \( DB \perp AC \)
   with \( B \) on \( AC \). \( BK \parallel AD \) with \( K \) on \( DC \)
   and \( DK : KC = 1 : 2 \).
   (a) Prove that:
      (1) \( BD^2 = AD \cdot BK \)
      (2) \( DA^2 = AB \cdot AC \)
   (b) Express in terms of \( BC \):
      (1) \( AB \)
      (2) \( AC \)
   (c) Hence prove that \( 2DA = \sqrt{3}BC \).
REVISION EXERCISE

1. In $\Delta ACE$, $BG \parallel CF$ and $AF = FE$. $AB = 2\text{cm}$, $BC = 3\text{cm}$, $CD = 1\text{cm}$ and $DE = 4\text{cm}$.

   (a) Determine $AG : GF$
   (b) Determine $\frac{EG}{GA}$
   (c) Prove that $DG \parallel CA$

2. O is the centre of the circle. Tangent $RQP$ meets chord $ST$ produced in $P$ such that $\hat{P} = 90^\circ$. $SO$ produced meets the tangent at $R$. $OR = 9\text{cm}$ and $OQ = 3\text{cm}$. Calculate, with reasons, the length of each of the following in simplest surd form:

   (a) $RQ$
   (b) $QP$

3. E and D are points on BC and AC of $\Delta ABC$ such that $\hat{DEC} = \hat{A}$. $AD = 9\text{ units}$, $DC = 15\text{ units}$ and $BC = 36\text{ units}$. $DF \parallel AB$ with $F$ on $BC$.

   (a) Prove that $\Delta CDE \parallel \Delta CBA$
   (b) Calculate:
       (1) $EC$
       (2) $CF$
       (3) $FE$

4. In $\Delta ABC$, $DE \parallel BC$, $FG \parallel DH$ and $\hat{ACB} = 90^\circ$. $AE = 2\text{cm}$ and $EC = 1\text{cm}$. Determine:

   (a) $\frac{AG}{GH}$
   (b) $\frac{DE}{BC}$
   (c) $\frac{AF}{FB}$
5. P, Q, R and S are points on a circle. QS and PR intersect at V. PT bisects QPR and Q₁ = Q₂.
Prove that:
(a) \( \hat{P}_3 = \hat{Q}_1 \)
(b) \( \hat{T}_1 = \hat{P}_2 + \hat{P}_3 \)
(c) \( \Delta PV S \parallel \Delta QPS \)
(d) \( PS^2 = SQ \cdot SV \)
(e) \( PS = ST \)
(f) \( \frac{PV}{Q} = \frac{VS}{ST} \)

6. AB is a diameter of circle ABCD. DE \perp AB and AC is joined.
Prove that:
(a) \( \hat{T}_1 = \hat{B} \)
(b) \( \hat{D}_1 = \hat{C}_1 \)
(c) \( DA^2 = AT \cdot AC \)

7. AB is a diameter of the circle ABDF centre O. \( \hat{A}_1 = \hat{E} \).
Prove that:
(a) CEDF is a cyclic quadrilateral.
(b) EA is a tangent to the circle at A.
(c) \( BE \cdot BD = BC \cdot BF \)

8. In \( \triangle KLM \), NO \parallel LM and NM and LO intersect at P. NP = 3 units, NO = 5 units, KN = 7 units and PM = 6 units.
Calculate, with reasons, the length of:
(a) \( LM \)
(b) \( KL \)
9. TA and TB are tangents to the circle centre O. AT produced meets OB produced at X. XB = 55 cm, TB = 48 cm and OA = r cm.
   (a) Prove that TAOB is a cyclic quadrilateral.
   (b) Calculate the lengths of TA and TX with reasons.
   (c) Calculate the length of r by using similar triangles.

10. In \( \triangle DEF \), PQ \parallel EF and SR = RQ.
    ER and PQ intersect at S.
    Prove that:
        (a) \( QF = SE \)
        (b) \( DP = \frac{DQ \cdot PE}{SE} \)

11. Circles OBC and OAD touch internally at O. EA is a tangent to the smaller circle and FOH is a tangent to the larger circle. EA || CO. Prove that:
    (a) AGCD is a parallelogram.
    (b) GAOC is a cyclic quadrilateral.
    (c) \( \frac{OD}{CD} = \frac{DA}{GB} \)

12. ABCD is a parallelogram. ABCE is a cyclic quadrilateral. Chords AF and BC intersect at G. EC is joined. DC is produced to F. Prove that:
    (a) \( AD = AF \)
    (b) \( EC = DC \)
    (c) \( DF = \frac{AD \cdot EC}{AG} \)

13. AB is a diameter of circle ABD. BD is produced to meet tangent AC at D.
    \( DC = \frac{1}{3} BC = x \).
    Prove that:
    \[ \sqrt{AD^2 + AB^2 + AC^2} = x\sqrt{11} \]
14. E and B are points on the sides AF and AC of \( \triangle ACF \) respectively such that BC = AE = 3 units, AB = 2 units and EF = 4.5 units. The circle through E, B and C cuts AF at D. FC produced meets the circle at H. EB, DC and BG are drawn. Prove that:

(a) \( BE \parallel CF \)  
(b) \( AB = \frac{BE \cdot AC}{CF} \)  
(c) \( \frac{AH \cdot BG}{HC} = \frac{BE \cdot AC}{CF} \)
MORE ADVANCED EXAMPLES

EXAMPLE 16

AB is a tangent to circle ACF at A. AC, CF and AF are chords. DE ⊥ AB. Prove that:

(a) CDFE is a cyclic quadrilateral.
(b) \(\triangle CDF \parallel \triangle AEB\)
(c) \(\frac{DA \cdot CG}{GH} = \frac{AE \cdot DF}{EB}\)

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) (\hat{A}_1 = \hat{C}_1) (\hat{A}_1 = \hat{E}_2) (\therefore \hat{C}_1 = \hat{E}_2) (\therefore \triangle ABCD) is a cyclic quadrilateral</td>
<td>tan-chord alt angles equal (\therefore \triangle CDE) subtends equal angles</td>
</tr>
<tr>
<td>(b) In (\triangle CDF \parallel \triangle AEB) : (1) (\hat{C}_1 = \hat{A}_1) (2) (\hat{D}_2 + \hat{D}_3 = \hat{E}_3) (3) (\hat{F}_2 = \hat{B}) (\therefore \triangle CDF \parallel \triangle AEB)</td>
<td>tan-chord ext angle of a cyclic quad sum of the angles of a triangle (\angle, \angle, \angle)</td>
</tr>
<tr>
<td>(c) (\frac{CD}{AE} = \frac{DF}{EB} = \frac{CF}{AB}) (\therefore \frac{CD}{AE} = \frac{DF}{EB}) (\therefore CD = \frac{AE \cdot DF}{EB}) But (\frac{CD}{DA} = \frac{CG}{GH}) (\therefore CD = \frac{DA \cdot CG}{GH}) (\therefore \frac{DA \cdot CG}{GH} = \frac{AE \cdot DF}{EB})</td>
<td>corr sides of (\triangle)’s in proportion Line (\parallel) one side of (\triangle)CAH</td>
</tr>
</tbody>
</table>
EXAMPLE 17
PN is a common tangent to the circles.
MRL is a tangent to the smaller circle.
KM, KN, KL and MN are chords.
RS is joined. Prove that:
(a) \( KR = \frac{MS \cdot RN}{SN} \)
(b) \( MR^2 = MS \cdot MN \)
(c) \( \frac{MR^2}{MN^2} = \frac{KR}{KN} \)

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) ( \hat{N}_1 = \hat{R}_1 ) ( \hat{N}_1 = \hat{R}_2 ) ( \therefore \hat{R}_1 = \hat{R}_2 ) ( \therefore ) ( KM \parallel RS ) ( \therefore \frac{KR}{RN} = \frac{MS}{SN} ) ( \therefore KR = \frac{MS \cdot RN}{SN} )</td>
<td>tan-chord tan-chord corr angles equal Line ( \parallel ) one side of ( \Delta KMN )</td>
</tr>
</tbody>
</table>
| (b) In \( \Delta MRS \) and \( \Delta MNR \):
(1) \( \hat{M}_2 = \hat{M}_2 \) | common |
(2) \( \hat{R}_2 = \hat{N}_2 \) | tan-chord |
(3) \( \hat{S}_2 = \hat{R}_1 + \hat{R}_2 \) | sum of the angles of a triangle \( \angle, \angle, \angle \) |
| \( \therefore \Delta MRS \parallel \Delta MNR \) | |
| \( \therefore \frac{MR}{MN} = \frac{RS}{NR} = \frac{MS}{MR} \) \( \therefore \frac{MR}{MN} = \frac{MS}{MR} \) \( \therefore \) \( MR^2 = MS \cdot MN \) | |
| (c) First simplify the ratio you are required to prove using previous parts of the question.
\( \frac{MR^2}{MN^2} = \frac{KR}{KN} \) \( \therefore \frac{MS \cdot MN}{MN^2} = \frac{KR}{KN} \) \( \therefore \frac{MS}{MN} = \frac{KR}{KN} \) | |
SOME CHALLENGES

1. A, B and C are concyclic.  
   BA produced meets the tangent through C at P. BC is produced to Q so that PQ = PC. 
   
   Prove that:
   
   (a) ACQP is a cyclic quadrilateral. 
   (b) \( PQ^2 = PA \cdot PB \)  
   (c) PQ is a tangent to circle ABQ.  
   (d) \( \frac{AC}{BC} = \frac{PC \cdot AP}{PQ^2} \)  

2. EBCD is a cyclic quadrilateral. 
   BE and CD produced cut at A. 
   BÇE = A.  
   
   Prove that:
   
   (a) \( \frac{BE}{BC} = \frac{CE}{AC} \)  
   (b) \( \frac{CE}{AC} = \frac{ED}{AD} \)
3. PA is a tangent to the circle ACBT. Chord BT II CA. Prove that:
   (a) \( PA^2 = PC \cdot PT \)
   (b) PT is a tangent to circle ADT.
   (c) 9DT \cdot DB = 4PT \cdot PC
       if \( AD = \frac{2}{3} AP \).

4. In the circle with centre O, AB is a diameter. CD is a tangent at D and FC bisects \( \hat{C} \).
   The radius \( r = 3 \) and \( CD = 4 \).
   (a) Prove that \( \triangle DBC \parallel \triangle ADC \)
   (b) Calculate the length of BC if \( BC = x \).
   (c) If \( \triangle DEC \parallel \triangle AFC \),
       show that \( CE = EF \).

5. BAD is a tangent to circle PAC. BPC is a straight line.
   (a) Prove that \( AB^2 = BP \cdot BC \)
   (b) Calculate BP if \( PC = 2 \) and \( AB = \sqrt{8} \).

6. AOB is a diameter of the circle ABC. DB is a tangent at B. ACD is a straight line. \( OB = BC = r \).
   Prove that:
   (a) \( \hat{B}_2 = 60^\circ \)
   (b) \( AC = \sqrt{3}r \)
   (c) \( AC:CD = 3:1 \)

7. AB is a chord of the circle centre O. AC \perp CB and AD = DB. OA and OB are joined.
   (a) \( \triangle OAD \parallel \triangle ABC \)
   (b) \( 2AD^2 = OA \cdot BC \).
8. ABC is a tangent to circle DPSGB. 
PS||ABC and DG||ABC. PG, PS, PB, 
SB and BG are joined. Chord BS and 
DG intersect at F. Chord PB and DG 
intersect at E. 
Prove that: 
(a) \( \hat{G}_1 = \hat{P}_1 \) 
(b) \( \triangle BGP || \triangle BEG \) 
(c) \( BG^2 = BP \cdot BE \) 
(d) \( \frac{BG^2}{BP^2} = \frac{BF}{BS} \)

9. AD is a tangent to circle BCD at D. 
CB is produced to A. CD and BD 
are chords. \( DC = \sqrt{2}DB \). 
Prove that: 
(a) \( \frac{DB}{CD} = \frac{AB}{AD} \) 
(b) \( AD^2 = AB \cdot AC \) 
(c) \( AB = BC \)

10. The tangent CP of circle ABC cuts 
BA produced at P such that \( BA = AP \). 
BC is produced to Q such that \( BP \perp PQ \) 
and \( PC = PQ \). 
Prove that: 
(a) \( AB \) is a diameter of the circle. 
(b) \( PC^2 = PA \cdot PB \) 
(c) \( BQ = \sqrt{6}PA \) 
(d) \( PQ = \sqrt{6}CA \)
1. The Comrades marathon is an ultramarathon of approximately 89 km which is run annually in KwaZulu-Natal between the cities of Durban and Pietermaritzburg. It is the world's largest and oldest ultramarathon race. The direction of the race alternates each year between the "up" run (87 km) starting from Durban and the "down" run (89 km) starting from Pietermaritzburg. The following table summarises the athletes who have achieved the most wins to date:

<table>
<thead>
<tr>
<th>Athlete</th>
<th>No of wins</th>
<th>Athlete</th>
<th>No of wins</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alan Robb</td>
<td>4</td>
<td>Helen Lucre</td>
<td>3</td>
</tr>
<tr>
<td>Arthur Newton</td>
<td>5</td>
<td>Jack Mekler</td>
<td>3</td>
</tr>
<tr>
<td>Bruce Fordyce</td>
<td>9</td>
<td>Lettie van Zyl</td>
<td>2</td>
</tr>
<tr>
<td>Elena Nurgalieva</td>
<td>7</td>
<td>Maria Bak</td>
<td>2</td>
</tr>
<tr>
<td>Frith van der Merwe</td>
<td>3</td>
<td>Maureen Holland</td>
<td>4</td>
</tr>
<tr>
<td>Hardy Ballington</td>
<td>5</td>
<td>Wally Hayward</td>
<td>5</td>
</tr>
</tbody>
</table>

(Source:http://en.wikipedia.org/wiki/Comrades_Marathon)

(a) Calculate the mean for this data.
(b) What is the mode for the number of wins per athlete?
(c) Draw a box and whisker plot for this data.
(d) Comment on the distribution of the data in terms of skewness.
(e) Calculate the standard deviation for this data (one decimal place).
(f) How many athletes lie outside the first standard deviation interval?
(g) Why is Bruce Fordyce’s number of wins an outlier? Give a mathematical reason to justify your answer.
(h) Draw a box and whisker plot for the data highlighting the outlier.

2. SA Idols is one of South Africa’s biggest television shows. It has been a huge success and the winner of this singing competition for 2012 was twenty-five year old Khaya Mthethwa, who was the favourite throughout the competition. The statistics for the competition from the Idols Top 10 phase appear below with each contestant’s number of votes represented by a percentage of the total.

Nospiko 3,86%
Obakeng 3,00%
Monde 3,89%
Chloe 5,80%
Simphiwe 9,01%
Dominic 3,40%
Melissa 3,29%
Khaya 33,09%
Shekhinah 9,64%
Tshidi 25,01%

(Source: http://idolssa.dstv.com)
(a) Calculate the mean percentage for this data.
(b) Draw a box and whisker plot for this data.
(c) Comment on the skewness of the data.
(d) Which of the percentages are outliers?
(e) Redraw the box and whisker plot highlighting the outliers.
(f) Calculate the standard deviation for the data.
(g) Which percentages lie outside the first standard deviation interval?

3. The box and whisker plots below represent the distances run by athletes from two different running clubs over a period of one month.

(a) What features are the same for both clubs?
(b) It seems that there is no significant difference in the performance between the two clubs. Is this conclusion valid? Support your answer with reasons.
(c) Comment on the distribution of distances for Club A if the mean is 76.
(d) Determine whether the minimum or maximum values for Club A are outliers.
(e) Does Club B have any outliers? Explain.

4. The following histogram represents the distance run by a few Comrades Marathon runners who didn’t complete the 89 km race.
(a) Redraw and complete the following table:

<table>
<thead>
<tr>
<th>Class interval</th>
<th>Frequency</th>
<th>Cumulative frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 &lt; x \leq 10$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$10 &lt; x \leq 20$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$20 &lt; x \leq 30$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$30 &lt; x \leq 40$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$40 &lt; x \leq 50$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$50 &lt; x \leq 60$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$60 \leq x \leq 70$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Calculate the estimated mean.
(c) State the modal class.
(d) Draw the cumulative frequency curve for this data.
(e) Determine estimates for the quartiles.
(f) Calculate the 60\textsuperscript{th} percentile.
(g) Draw a box and whisker diagram if the minimum distance is 10 km and the maximum distance is 70 km.
(h) Comment on the distribution of the data in terms of skewness.

5. Residents in Knysna complained to the Police department that too many drivers were exceeding the speed limit of 60 km/h on a busy road near a school. The Police department decided to investigate the situation. The department recorded the speeds of drivers along this road over one week. The speed limit is 60 km/h. The following cumulative frequency curve represents the findings of the Department.
(a) How many drivers were there in total?
(b) How many drivers exceeded the speed limit of 60 km/h?
(c) How many drivers were within the speed limit?
(d) What is the median speed for this data?
(e) How many drivers had a speed of less than 80 km/h?
(f) What percentage of drivers had a speed above 100 km/h?
(g) Complete the following table:

<table>
<thead>
<tr>
<th>Speed in km/h (x)</th>
<th>Frequency</th>
<th>Cumulative frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>30 ≤ x &lt; 40</td>
<td></td>
<td>12</td>
</tr>
<tr>
<td>40 ≤ x &lt; 50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50 ≤ x &lt; 60</td>
<td></td>
<td></td>
</tr>
<tr>
<td>60 ≤ x &lt; 70</td>
<td></td>
<td></td>
</tr>
<tr>
<td>70 ≤ x &lt; 80</td>
<td></td>
<td></td>
</tr>
<tr>
<td>80 ≤ x &lt; 90</td>
<td></td>
<td></td>
</tr>
<tr>
<td>90 ≤ x &lt; 100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100 ≤ x &lt; 110</td>
<td></td>
<td></td>
</tr>
<tr>
<td>110 ≤ x &lt; 120</td>
<td></td>
<td></td>
</tr>
<tr>
<td>120 ≤ x ≤ 130</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(h) Calculate an estimated mean speed.
(i) Calculate an estimated value for the standard deviation of this data.
   (Hint: Use the midpoints of the class intervals as the x-values to enter in your calculator)

4. A medical researcher recorded the growth in the number of bacteria over a period of 10 hours. The results are recorded in the following table:

<table>
<thead>
<tr>
<th>Time in hours</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of bacteria</td>
<td>5</td>
<td>10</td>
<td>75</td>
<td>13</td>
<td>10</td>
<td>20</td>
<td>30</td>
<td>35</td>
<td>45</td>
<td>65</td>
<td>80</td>
<td>1</td>
</tr>
</tbody>
</table>

(a) Draw a scatterplot to represent this data.
(b) State the type of relationship (linear, quadratic or exponential) that exists between the number of hours and the growth in the number of bacteria.
(c) Are there any outliers? Explain possible causes for these outliers.
SCATTERPLOTS

Scatterplots are useful diagrams that enable researchers to compare two sets of data to determine whether there is a relationship between them. For example, the relationship between a smoking habit and the chance of getting lung cancer can be investigated using a scatterplot diagram. The number of cars on a highway and the accident rate are two sets of data that might have a relationship or correlation. Plotting this data on a scatterplot diagram will show trends in the data. Data could follow a linear, quadratic or exponential trend.

LINES OF BEST FIT

When data follows a linear trend, a line of best fit can be drawn on the scatterplot diagram. The following example will illustrate what the line of best fit is.

EXAMPLE 1

Consider the following scatterplot of information obtained by a publishing company that recorded the number of books sold over a period of 12 months. Investigate the relationship between the two sets of data: number of months and the number of books sold.

<table>
<thead>
<tr>
<th>No. of months</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of books sold</td>
<td>10</td>
<td>22</td>
<td>20</td>
<td>38</td>
<td>46</td>
<td>100</td>
<td>48</td>
<td>62</td>
<td>61</td>
<td>74</td>
<td>88</td>
<td>86</td>
</tr>
</tbody>
</table>

![Scatterplot diagram with data points and lines of best fit for linear, quadratic, and exponential trends.]
It is clear from the above graph that the points do not lie on a perfect straight line. The information on the graph does follow a linear trend. We can draw what is called a **line of best fit**, which will help to predict future values. In order to do this, draw a line through some of the points with the aim of having the same number of points above the line as below the line. A possible line of best fit will now be drawn onto the graph.

A line can be drawn through two of the points with five above and five below. A line of best fit will not represent the data perfectly, but it will give you an idea of the trend. Different people will probably draw slightly different lines of best fit, but the trend should be more or less the same in each case. One of the points (see diagram) is way out compared to the others. This point is called an **outlier**. Probably what happened during the sixth month was that the company had an unusually high sale of books. There might have been a conference where the company displayed and sold a lot of books. What is evident though, is that the number of books sold has a linear increase over the 12 months.

The line of best fit drawn in the diagram above is not really accurate since different people will have drawn different lines of best fit. What we will now demonstrate is a method to determine the actual equation of the line of best fit using the least squares method.

The least squares method involves determining the gradient and y-intercept of the line of best fit by using a calculator. The line of best fit is also called a **regression line**. The equation of this regression line (line of best fit) is given by \( y = a + bx \) where \( a \) represents the y-intercept of the line and \( b \) the gradient or slope. A calculator can be used to determine the value of \( a \) and \( b \). The outlier will need to be excluded as it is too far away from the line of best fit.

A possible calculator programme is as follows:

```
MODE 2
PRESS: 2 : A + BX
ENTER DATA POINTS:
COLUMN (X)
1= 2= 3= 4= 5= 7= 8= 9= 10= 11= 12=
COLUMN (Y)
10 22 20 38 46 48 62 61 74 88 86
```
THEN PRESS AC
THEN PRESS SHIFT 1
THEN PRESS 5: REG
THEN PRESS 1: A = TO GET THE VALUE OF a WHICH IS 5,315923567
THEN PRESS SHIFT 1
THEN PRESS 5: REG
THEN PRESS 2: B = TO GET THE VALUE OF b WHICH IS 6,896178344
THEN PRESS AC
THEN PRESS MODE 1 TO GET BACK TO NORMAL MODE
\[ y = 6,896178344x + 5,315923567 \] is the equation of the line of best fit for this data, where the y-intercept is 5,315923567 and the gradient is 6,896178344.

A very important principle is that the line of best fit passes through the mean point: 
\[ (\bar{x}; \bar{y}) \]
\[ \bar{x} = \frac{1 + 2 + 3 + 4 + 5 + 7 + 8 + 9 + 10 + 11 + 12}{11} = 6,54545454 \approx 6,5 \]
\[ \bar{y} = \frac{10 + 22 + 20 + 38 + 46 + 48 + 62 + 61 + 74 + 88 + 86}{11} = 50,4545454 \approx 50,5 \]
The line of best fit therefore passes through the point (6,5;50,5)

In order to draw the line of best fit, plot the y-intercept and mean point and draw the line through these points.

Notice that if the outlier is included, the line of best fit will be “pulled upwards” and will not be accurate.
The line will have an equation of 
\[ y = 6,70979021x + 10,96969697 \] and the mean point will be (6,5;55)
It is now possible to use the line of best fit to predict an estimate of the number of books that might be sold for the 13th month:
\[ y = 6,896178344(13) + 5,315923567 = 94 \text{ books} \]

This process of estimating the number of books outside the given 12 months is called **extrapolation**. If the line of best fit is used to estimate the number of books inside the given 12 months, then this is called **interpolation**. For example, an estimate of the number of books sold after ten and a half months can be done as follows and is an example of interpolation:
\[ y = 6,896178344(10.5) + 5,315923567 = 77 \text{ books} \]

Remember that this linear increase over the year might not necessarily continue in this manner. If the sales start dropping in the near future, the trend will no longer be a linear increase. It could change to a linear decrease (or even a quadratic trend) if the sales of books start dropping.

**EXERCISE 1**
(Photocopiable grids are provided in the Teacher guide)

1. The table below represents the distance in metres required by a car to apply brakes and reach a standstill when it is travelling at a given speed.

<table>
<thead>
<tr>
<th>Speed km/h</th>
<th>Braking distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>6</td>
</tr>
<tr>
<td>40</td>
<td>16</td>
</tr>
<tr>
<td>60</td>
<td>30</td>
</tr>
<tr>
<td>80</td>
<td>48</td>
</tr>
<tr>
<td>100</td>
<td>70</td>
</tr>
<tr>
<td>120</td>
<td>80</td>
</tr>
<tr>
<td>140</td>
<td>110</td>
</tr>
</tbody>
</table>

(a) Draw a scatter plot to represent this data.
(b) Explain whether a linear, quadratic or exponential curve would be a line or curve of best fit.
(c) Determine the equation of the regression line.
(d) Draw the regression line on the scatterplot diagram.
(e) Use your regression line to estimate the breaking distance at a speed of: (1) 150 km/h (2) 130 km/h

2. The table below shows the acidity of eight dams near an industrial plant in Limpopo and their distance from it.

<table>
<thead>
<tr>
<th>Distance (km)</th>
<th>4</th>
<th>34</th>
<th>17</th>
<th>60</th>
<th>6</th>
<th>52</th>
<th>42</th>
<th>31</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acidity (pH)</td>
<td>3.0</td>
<td>4.4</td>
<td>3.2</td>
<td>7.0</td>
<td>3.2</td>
<td>6.8</td>
<td>5.2</td>
<td>4.8</td>
</tr>
</tbody>
</table>

(a) Draw a scatterplot to represent this data.
(b) Determine the equation of the regression line.
(c) Draw a line of best fit on the diagram.
(d) Use your line of best fit to predict the acidity of the dam at a distance of 25 kilometres.
3. (a) A leading nursery in Johannesburg recorded the effect of temperature on the growth of a new plant that has recently been imported into the country. The goal of the study is to determine what temperature is ideal for maximum flowering for a particular plant.

<table>
<thead>
<tr>
<th>Temperature</th>
<th>Number of flowers</th>
</tr>
</thead>
<tbody>
<tr>
<td>25°C</td>
<td>2</td>
</tr>
<tr>
<td>26°C</td>
<td>3</td>
</tr>
<tr>
<td>27°C</td>
<td>4</td>
</tr>
<tr>
<td>28°C</td>
<td>6</td>
</tr>
<tr>
<td>29°C</td>
<td>7</td>
</tr>
<tr>
<td>30°C</td>
<td>7</td>
</tr>
<tr>
<td>31°C</td>
<td>8</td>
</tr>
<tr>
<td>32°C</td>
<td>9</td>
</tr>
<tr>
<td>33°C</td>
<td>10</td>
</tr>
<tr>
<td>34°C</td>
<td>14</td>
</tr>
</tbody>
</table>

(1) Draw a scatter plot to represent this bivariate data.
(2) Describe the trends shown by this scatterplot.
(3) Determine the equation of the regression line (called the line of best fit).
(4) Predict the number of flowers if the temperature is 38°C.

(b) From 39°C upwards the number of flowers starts to decrease. The data is represented in the table below.

<table>
<thead>
<tr>
<th>Temperature</th>
<th>Number of flowers</th>
</tr>
</thead>
<tbody>
<tr>
<td>39°C</td>
<td>15</td>
</tr>
<tr>
<td>40°C</td>
<td>8</td>
</tr>
<tr>
<td>41°C</td>
<td>6</td>
</tr>
<tr>
<td>42°C</td>
<td>5</td>
</tr>
<tr>
<td>43°C</td>
<td>3</td>
</tr>
</tbody>
</table>

(1) Draw a scatter plot to represent this bivariate data.
(2) Determine the equation of the regression line.
(3) Draw the line of best fit on the scatterplot diagram.
(4) Predict the temperature at which the plant will no longer produce any more flowers.

4. A Gauteng motor company did research on how the speed of a car affects the fuel consumption of the vehicle. The following data was obtained:

<table>
<thead>
<tr>
<th>Speed in km/h</th>
<th>60</th>
<th>75</th>
<th>115</th>
<th>85</th>
<th>110</th>
<th>95</th>
<th>120</th>
<th>100</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fuel consumption in ℓ/100 km</td>
<td>11,5</td>
<td>10</td>
<td>8,4</td>
<td>9,2</td>
<td>7,8</td>
<td>8,9</td>
<td>8,8</td>
<td>8,6</td>
<td>10,2</td>
</tr>
</tbody>
</table>

(a) Draw a scatter plot to represent this bivariate data.
(b) Suggest whether a linear, quadratic or exponential function would best fit the data.
(c) What advice can the company give about the driving speed in order to keep the cost of fuel to a minimum?
5. In a recent article published in 2012 by a leading South African insurance company, the following was stated: "South Africa has one of the highest per capita alcohol consumption rates in the world, with over 30% of the population struggling with an alcohol problem or on the verge of having one. When you are on the road at night, one out of every seven drivers on the road with you is drunk. Drunk drivers cause accidents and snuff out innocent lives on the country’s roads almost daily, indirectly affecting numerous lives in the process. In 2009, the World Health Organisation reported that 60% of road traffic deaths in South Africa involve alcohol, while the South African Medical Research Council suggests that alcohol is a factor in 50% of all road accidents. In South Africa, the legal blood alcohol level is 0.05 grams per 100 millilitres and the legal breath alcohol level is 0.24 milligrams per 1000 millilitres. For someone of average weight, just two or three glasses of wine are enough to push him/her well over the legal limit". [Source: www.psypress.com]

A researcher interested in the relationship between blood alcohol levels and road accidents examined the number of accidents for various blood alcohol levels in certain towns in South Africa during 2012. The results are recorded in the following table:

<table>
<thead>
<tr>
<th>Alcohol level (g/100ml)</th>
<th>0.005</th>
<th>0.010</th>
<th>0.015</th>
<th>0.020</th>
<th>0.025</th>
<th>0.030</th>
<th>0.035</th>
<th>0.040</th>
<th>0.045</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of accidents</td>
<td>100</td>
<td>170</td>
<td>260</td>
<td>300</td>
<td>320</td>
<td>380</td>
<td>420</td>
<td>450</td>
<td>500</td>
</tr>
<tr>
<td>Alcohol level (g/100ml)</td>
<td>0.050</td>
<td>0.055</td>
<td>0.060</td>
<td>0.065</td>
<td>0.070</td>
<td>0.075</td>
<td>0.080</td>
<td>0.085</td>
<td>0.090</td>
</tr>
<tr>
<td>No. of accidents</td>
<td>550</td>
<td>570</td>
<td>590</td>
<td>900</td>
<td>600</td>
<td>630</td>
<td>650</td>
<td>625</td>
<td>700</td>
</tr>
</tbody>
</table>

(a) Draw a scatterplot to represent this bivariate data.
(b) What data value is an outlier? Explain possible reasons for this outlier.
(c) Determine the equation of the regression line. Remember to exclude the outlier.
(d) Draw the line of best fit on the scatterplot diagram.
(e) Predict the possible number of accidents if the blood alcohol level is 0.1.
(f) Government wants to drop the legal limit to 0.02. Do you think that is justifiable based on the findings of the research?

6. **HIV/AIDS** is a major public health concern and the cause of death in many parts of the world. Although Africa is home to about 14.5% of the world's population, it is estimated to be home to 69% of all people living with HIV. Southern Africa is the worst affected region of Africa, as well as the worst affected region in the world, with the epidemic reaching very high levels in Swaziland, Botswana, Lesotho, South Africa, Zimbabwe and Namibia. The percentages of deaths in South Africa that were caused by AIDS (from
the years 2001 – 2011) are recorded in the table below and a scatterplot diagram has been drawn. (Source: www.statsa.gov.za/publications)

<table>
<thead>
<tr>
<th>Year</th>
<th>Total number of deaths</th>
<th>Total number of AIDS deaths</th>
<th>Percentage AIDS deaths</th>
</tr>
</thead>
<tbody>
<tr>
<td>2001</td>
<td>532 482</td>
<td>215 907</td>
<td>41%</td>
</tr>
<tr>
<td>2002</td>
<td>577 444</td>
<td>259 043</td>
<td>45%</td>
</tr>
<tr>
<td>2003</td>
<td>618 293</td>
<td>298 297</td>
<td>48%</td>
</tr>
<tr>
<td>2004</td>
<td>652 868</td>
<td>331 794</td>
<td>51%</td>
</tr>
<tr>
<td>2005</td>
<td>678 386</td>
<td>356 209</td>
<td>53%</td>
</tr>
<tr>
<td>2006</td>
<td>676 660</td>
<td>353 577</td>
<td>52%</td>
</tr>
<tr>
<td>2007</td>
<td>664 009</td>
<td>339 666</td>
<td>51%</td>
</tr>
<tr>
<td>2008</td>
<td>640 521</td>
<td>315 103</td>
<td>49%</td>
</tr>
<tr>
<td>2009</td>
<td>611 338</td>
<td>283 437</td>
<td>46%</td>
</tr>
<tr>
<td>2010</td>
<td>593 907</td>
<td>263 368</td>
<td>44%</td>
</tr>
<tr>
<td>2011</td>
<td>591 366</td>
<td>257 910</td>
<td>43%</td>
</tr>
</tbody>
</table>

(a) Describe the trends for this data.
(b) Explain the possible reasons for the decrease in the number of deaths caused by AIDS from 2005 until 2011.
(c) What do you think needs to be done in South Africa in the future to keep bringing down the deaths caused by HIV/AIDS?

7. **Obesity** has become a global epidemic with an estimated 1.3 billion people overweight or obese. In the United States, it is as high as 27% in men and 32% in women above 20 years. Obesity negatively affects the lives of many South Africans and the consequent burden of disease contributes to the increasing cost of health care, both at a state level and in the private sector. The measure adopted for being obese is referred to as the body mass index (BMI).
Here are the definitions of BMI, overweight and obese:

\[
\text{BMI} = \frac{\text{mass in kg}}{\text{(height in m)}^2}
\]

Overweight is a BMI of between 25 and 29.9. Obesity is defined as a BMI of greater than 30. The following table summarises the percentages of South Africans in four different BMI categories for 1998. Scatterplots are provided as well. The percentages for obesity have increased over the years.

<table>
<thead>
<tr>
<th>Age group</th>
<th>Underweight BMI &lt; 18.5</th>
<th>Normal 18.5 &lt; BMI &lt; 24.9</th>
<th>Overweight 25 &lt; BMI &lt; 29.9</th>
<th>Obese BMI &gt; 30</th>
</tr>
</thead>
<tbody>
<tr>
<td>15-24</td>
<td>9.6</td>
<td>60.7</td>
<td>20.0</td>
<td>9.6</td>
</tr>
<tr>
<td>25-34</td>
<td>5.1</td>
<td>38.4</td>
<td>29.2</td>
<td>27.0</td>
</tr>
<tr>
<td>35-44</td>
<td>2.7</td>
<td>27.2</td>
<td>30.7</td>
<td>39.3</td>
</tr>
<tr>
<td>45-54</td>
<td>3.7</td>
<td>23.9</td>
<td>26.5</td>
<td>45.5</td>
</tr>
<tr>
<td>55-64</td>
<td>2.7</td>
<td>25.6</td>
<td>25.6</td>
<td>46.1</td>
</tr>
<tr>
<td>65+</td>
<td>7.4</td>
<td>32.5</td>
<td>26.5</td>
<td>33.3</td>
</tr>
</tbody>
</table>

(a) Which age group has the highest percentage of obese South Africans?
(b) Which BMI category has the closest percentages across the age groups?
(c) Describe the percentage trends for each of the four BMI categories across the different age groups. What type of function, where appropriate, fits the data for each category?

(d) What factors contribute to increasing percentages of obesity in South Africa? Google search obesity on the Internet for relevant information.

(e) What are some of the major health risks associated with obesity?

(f) What measures should South Africans take in order to reduce or prevent obesity?

**CORRELATION**

The strength of the linear relationship between the two variables in a scatterplot depends on how close the data points are to the line of best fit. The closer the points are to this line, the stronger the relationship. If the points are further away from the line of best fit, the weaker the relationship. If the line of best fit slopes to the right and has a positive gradient, then the linear relationship is positive. If the line of best fit slopes to the left and has a negative gradient, then the linear relationship is negative.

The gradient of the line of best fit indicates whether the association is positive or negative. However, it will not indicate the strength of the association. The correlation coefficient \( r \) is a value that gives an indication of the strength of the association. It can be calculated using a calculator.

**Strong positive linear association**

\[ r > 0 \text{ and very close to 1} \]

**Moderately positive linear association**

\[ r > 0 \text{ and fairly close to 1} \]

**Perfect positive linear association**

\[ r = 1 \]

**No positive correlation**

\[ r = 0 \]

**Strong negative linear association**

\[ r < 0 \text{ and very close to } -1 \]

**Moderate negative linear association**

\[ r < 0 \text{ and fairly close to } -1 \]
Perfect negative linear association  
\[ r = -1 \]

No negative correlation  
\[ r = 0 \]

The following table provides different examples of \( r \) and how to interpret these values.

<table>
<thead>
<tr>
<th>( r )</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Perfect positive association</td>
</tr>
<tr>
<td>0.9</td>
<td>Strong positive association</td>
</tr>
<tr>
<td>0.5</td>
<td>Moderate positive association</td>
</tr>
<tr>
<td>0.2</td>
<td>Weak positive association</td>
</tr>
<tr>
<td>0</td>
<td>No association</td>
</tr>
<tr>
<td>-0.2</td>
<td>Weak negative association</td>
</tr>
<tr>
<td>-0.5</td>
<td>Moderate negative association</td>
</tr>
<tr>
<td>-0.9</td>
<td>Strong negative association</td>
</tr>
<tr>
<td>-1</td>
<td>Perfect negative association</td>
</tr>
</tbody>
</table>

EXAMPLE 2

The following table shows information about the age of eight fossils of a particular species of ape found in South Africa and the interior volume of their skulls.

<table>
<thead>
<tr>
<th>Age in millions of years</th>
<th>2.4</th>
<th>2.0</th>
<th>0.8</th>
<th>1.1</th>
<th>1.6</th>
<th>0.7</th>
<th>0.3</th>
<th>1.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volume of skull (cm³)</td>
<td>38</td>
<td>82</td>
<td>174</td>
<td>171</td>
<td>89</td>
<td>159</td>
<td>206</td>
<td>129</td>
</tr>
</tbody>
</table>

Calculate the correlation coefficient (\( r \)) and comment on the strength of the linear association.

Solution

A possible calculator programme is as follows:
MODE 2
PRESS 2: A + BX
ENTER DATA POINTS:
IN AGE COLUMN (X)
2.4 = 2.0 = 0.8 = 1.1 = 1.6 = 0.7 = 0.3 = 1.4 =

IN COLUMN (Y)
38 = 82 = 174 = 171 = 89 = 159 = 206 = 129 =
THEN PRESS AC
THEN PRESS SHIFT 1
THEN PRESS 5: REG
THEN PRESS 3: $r = \text{TO GET THE VALUE OF } r \text{ WHICH IS } r = -0.9679973645$
THEN PRESS MODE 1 TO GET BACK TO NORMAL MODE

Hence $r = -0.9679973645$

Therefore there is a strong negative linear association between the variables.

**EXERCISE 2**

1. The table shows the pH level of eight dams near to an industrial complex and their distances from it. Calculate the correlation coefficient and comment on the strength of the linear association.

<table>
<thead>
<tr>
<th>Distance in km</th>
<th>5</th>
<th>35</th>
<th>18</th>
<th>7</th>
<th>61</th>
<th>53</th>
<th>43</th>
<th>57</th>
</tr>
</thead>
<tbody>
<tr>
<td>pH level</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>7</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

2. The table shows the number of kilometers still to go after different ten-minute periods. Calculate the correlation coefficient and comment on the strength of the linear association.

<table>
<thead>
<tr>
<th>Time in minutes</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance in km</td>
<td>72</td>
<td>65</td>
<td>54</td>
<td>41</td>
<td>48</td>
<td>35</td>
<td>24</td>
<td>18</td>
</tr>
</tbody>
</table>

3. The table below records the results of a study investigating how the number of cigarettes smoked per day is related to the number of people contracting lung cancer. Calculate the correlation coefficient and comment on the strength of the linear association.

<table>
<thead>
<tr>
<th>Number of cigarettes per day</th>
<th>Number of cases of lung cancer</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>15</td>
<td>9</td>
</tr>
<tr>
<td>20</td>
<td>18</td>
</tr>
<tr>
<td>25</td>
<td>20</td>
</tr>
<tr>
<td>30</td>
<td>26</td>
</tr>
</tbody>
</table>
4. The table below records the results of a study investigating how the number of times exercised per week reduces the number of stress related headaches. Calculate the correlation coefficient and comment on the strength of the linear association.

<table>
<thead>
<tr>
<th>Number of days exercised</th>
<th>Number of headaches</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>

5. The table below records the results of a study investigating the effect on the number of apples eaten per day with success in Mathematics. Determine the correlation coefficient and comment on the strength of the linear association.

<table>
<thead>
<tr>
<th>Number of apples eaten</th>
<th>Results in Maths</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>45%</td>
</tr>
<tr>
<td>2</td>
<td>67%</td>
</tr>
<tr>
<td>3</td>
<td>23%</td>
</tr>
<tr>
<td>4</td>
<td>53%</td>
</tr>
<tr>
<td>5</td>
<td>39%</td>
</tr>
<tr>
<td>6</td>
<td>41%</td>
</tr>
</tbody>
</table>

6. Calculate the correlation coefficient for each of the following and comment on the strength of the linear association:
   (a) Exercise 1 number 1
   (b) Exercise 1 number 2
   (c) Exercise 1 number 3(a)
   (d) Exercise 1 number 3(b)
   (e) Exercise 1 number 5
REVISION EXERCISE

1. During the Gauteng winter month of July, a number of patients visited a local hospital suffering from influenza. The table below shows the number of patients treated.

<table>
<thead>
<tr>
<th>Dates in the month of July</th>
<th>4</th>
<th>6</th>
<th>10</th>
<th>12</th>
<th>16</th>
<th>20</th>
<th>22</th>
<th>26</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of patients treated</td>
<td>260</td>
<td>280</td>
<td>380</td>
<td>420</td>
<td>600</td>
<td>680</td>
<td>800</td>
<td>820</td>
</tr>
</tbody>
</table>

(a) Draw a scatterplot of the above data.
(b) Determine the equation of the least squares regression line for the data.
(c) Draw the line of best fit on your scatterplot diagram.
(d) Estimate how many patients were treated as at 28 June.
(e) Estimate how many patients were treated as at 23 July.
(f) Estimate how many patients were treated as at 31 July.
(g) Calculate the correlation coefficient for the data. Interpret this result.
CHAPTER 10 – PROBABILITY

SUMMARY OF ALL PROBABILITY THEORY (GRADE 10 AND 11)

1. Consider the probability rule
   \[ P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \]
   A and B are **inclusive** if \( P(A \text{ and } B) \neq 0 \) (there is an intersection)

   ![Inclusive Events](image)

   A and B are **mutually exclusive** if \( P(A \text{ and } B) = 0 \) (there is no intersection)

   ![Mutually Exclusive Events](image)

   For mutually exclusive events \( P(A \text{ or } B) = P(A) + P(B) \)

2. **Exhaustive events** contain all elements of the sample space between them.
   In this case, \( P(A \text{ or } B) = 1 \)

   ![Exhaustive Events](image)

3. **Complementary events** are mutually exclusive and exhaustive.
   In this case, \( P(A) + P(B) = 1 \)

   ![Complementary Events](image)

   The set \{not A\} is B since all elements that are not in A are in B.
   We can write \( \text{not } A = B = A^l \)

4. Where events A and B are not complementary, then the set \{not A\} will be different to B.
   \( B = \{c ; d ; e ; f\} \) and not A = \( \{c ; d ; e ; f ; g\} \)

   ![Not A](image)

5. For **independent events**, \( P(A \text{ and } B) = P(A) \times P(B) \)
REVISION EXERCISE

1. Consider the following statements:
   A. \( P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \) where \( P(A \text{ and } B) \neq 0 \)
   B. \( P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \) where \( P(A \text{ and } B) = 0 \)
   C. \( P(A \text{ or } B) = 1 \)
   D. \( P(A \text{ or } B) \neq 1 \)
   E. \( P(A) + P(B) = 1 \)
   F. complementary
   G. inclusive
   H. mutually exclusive

Indicate which of the above statements apply to each of the following Venn diagrams.

(a) ![Venn Diagram A]
(b) ![Venn Diagram B]
(c) ![Venn Diagram C]
(d) ![Venn Diagram D]

2. A six-sided die is rolled and the number of dots landing face up is noted. Consider the following events:
   \( A = \{ \text{numbers less than 3} \} \) \quad \( B = \{ \text{even numbers} \} \) \quad \( C = \{ \text{the number 4} \} \)
   (a) What is the sample space?
   (b) Determine \( P(A) \), \( P(B) \) and \( P(C) \).
   (c) Determine \( P(A^c) \).
   (d) Are the events mutually exclusive? Give a reason.
   (e) Determine \( P(A \text{ or } C) \).
   (f) Determine \( P(A \text{ or } B) \).
   (g) Are events \( B \) and \( C \) independent? Give a reason.

3. Mpho writes a Science and a Maths examination. He believes that he has a 40\% chance of passing the Maths examination, a 60\% chance of passing the Science examination and a 30\% chance of passing both. What is the probability that he will pass at least one subject?

4. In a Science quiz, two teams work independently on a problem. They are allowed a maximum of 15 minutes to solve the problem. The probabilities that each team will solve the problem are \( \frac{1}{2} \) and \( \frac{1}{3} \) respectively. Calculate the probability that the problem will be solved in the 15 minutes allowed.
5. \[ P(A) = \frac{3}{10} \quad \text{and} \quad P(B) = \frac{1}{2} \]. Calculate \( P(A \text{ or } B) \) if:

(a) \( A \) and \( B \) are mutually exclusive events.
(b) \( A \) and \( B \) are independent events.

6. There are 30 dogs and 25 cats at a pet shop. The owner chooses individual pets at random and sells them.

(a) Calculate the probability that the first pet chosen is a dog.
(b) Draw a tree diagram to represent the situation if the owner chooses three pets, one after the other.
   Indicate on your diagram all possible outcomes.
(c) Calculate the probability that a dog, then a cat and then another dog is chosen in that order.
(d) Calculate the probability that all three pets chosen are cats.
(e) Calculate the probability that at least one of the pets chosen is a dog.

7. In a factory three machines, \( A \), \( B \) and \( C \), manufacture steel bolts. They produce 20\%, 35\% and 45\% of the total production respectively. 16\%, 6\% and 3\% of the bolts produced by \( A \), \( B \) and \( C \) respectively are defective.

(a) Draw a tree diagram.
(b) Calculate the probability that a bolt is manufactured by machine \( A \) and is defective.
(c) Calculate the probability that a selected bolt is defective.

8. At a school for boys there are 240 learners in Grade 12. The following information was gathered about participation in school sport:
   120 boys play rugby (R)
   60 boys play soccer (B)
   95 boys play cricket (C)
   17 boys play all three sports
   23 boys play rugby and soccer
   27 boys play cricket and soccer
   26 boys do not play any of these sports

Let the number of learners who play rugby and cricket only be \( x \).

(a) Draw a Venn diagram to represent the above information.
(b) Determine the number of boys who play rugby and cricket.
(c) Determine the probability that a learner in Grade 12 selected at random:
   (1) only plays soccer.
   (2) does not play cricket.
   (3) participates in at least two of these sports.
THE FUNDAMENTAL COUNTING PRINCIPLE

Consider the situation where you flip a coin and then roll a die. There are 12 possible outcomes in this situation:

\[
\begin{array}{ccccccc}
H 1 & H 2 & H 3 & H 4 & H 5 & H 6 \\
T 1 & T 2 & T 3 & T 4 & T 5 & T 6 \\
\end{array}
\]

For the coin there are only two possible outcomes, whereas for the die, there are six possible outcomes. Therefore we can state that there is a total of \(2 \times 6 = 12\) total possible outcomes.

This shortcut approach is called The Fundamental Counting Principle and can be stated in general as follows:

**RULE 1**

If one operation can be done in \(m\) ways and a second operation can be done in \(n\) ways then the total possible number of different ways in which both operations can be done is \(m \times n\).

**EXAMPLE 1**

A meal can be made up as follows:
Choice 1: meat, fish or chicken
Choice 2: mash, chips, baked potato, rice or vegetables

How many different meals can be made using these choices?

**Solution**

There are 3 possible choices in Choice 1 followed by 5 possible choices in Choice 2. There are therefore 15 possible meals that can be made:

<table>
<thead>
<tr>
<th>meat, mash</th>
<th>fish, mash</th>
<th>chicken, mash</th>
</tr>
</thead>
<tbody>
<tr>
<td>meat, chips</td>
<td>fish, chips</td>
<td>chicken, chips</td>
</tr>
<tr>
<td>meat, baked potato</td>
<td>fish, baked potato</td>
<td>chicken, baked potato</td>
</tr>
<tr>
<td>meat, rice</td>
<td>fish, rice</td>
<td>chicken, rice</td>
</tr>
<tr>
<td>meat, vegetables</td>
<td>fish, vegetables</td>
<td>chicken, vegetables</td>
</tr>
</tbody>
</table>

According to the fundamental counting principle, there are \(3 \times 5 = 15\) possible meals that can be made.

**EXAMPLE 2**

A gift pack can be made up as follows:
Choice 1: Choose one CD out of a possible 4 different CD’s
Choice 2: Choose one packet of chips out of a possible 5 different types
Choice 3: Choose one chocolate type out of a possible 12 different types
Choice 4: Choose one type of fruit out of a possible 3 different fruit types

How many different gift packs can be made?

**Solution**

\[4 \times 5 \times 12 \times 3 = 720\] different gift packs
EXAMPLE 3

Consider the word PARKTOWN. You are required to form different eight-letter word arrangements using the letters of the word PARKTOWN. An example of a word arrangement would be the word APKROTWN. This arrangement of the letters need not make any sense. How many possible word arrangements can be made if:
(a) the letters may be repeated?
(b) the letters may not be repeated?

Solutions

(a) An example of an eight-letter word arrangement where the letters may be repeated is AAPKWANO. This means that for the first letter of the word arrangement, any of the eight letters in the word PARKTOWN can be used. For the second letter, any of the eight letters may be used again. In the word arrangement AAPKWANO, the letter A is used as the first letter and then again as the second letter as well as the sixth letter. In forming a word arrangement, there are 8 possible letters that can be used for the first letter. For the second letter, we can still use 8 letters (repeating the letters is allowed). For the third letter there are still 8 possible letters available to use. From the fundamental counting principle, there are:
\[ 8 \times 8 \times 8 \times 8 \times 8 \times 8 = 8^8 = 16,777,216 \] word arrangements.

(b) An example of an eight-letter word arrangement where the letters may not be repeated is NOTWARPK. This means that for the first letter of the word arrangement, any of the eight letters in the word PARKTOWN can be used. However, for the second letter, only seven letters may be used since the first letter may not be used again. In the word arrangement NOTWARPK, the letter N is used as the first letter but not again as the second letter. In forming a word arrangement, there are 8 possible letters that can be used for the first letter. For the second letter, we can use 7 letters (repeating the letters is not allowed). For the third letter, there will be 6 possible letters available to use. For the eighth letter, there will only be one choice. This will be the last remaining letter that was not used. From the fundamental counting principle, there are:
\[ 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40,320 \] word arrangements.

EXAMPLE 4

How many different ways are there of predicting the results of 5 rugby matches where each match can end in either a win, lose or draw?

Solution

For each match, there are 3 possible outcomes: win, lose or draw. So for a total of 5 matches there are:
\[ 3 \times 3 \times 3 \times 3 \times 3 = 3^5 = 243 \] possible results
EXERCISE 1

1. A party pack of three items can be made up by selecting one item from each of the following choices:
   - Choice 1: Smarties, Astros, Jelly Tots, Wine Gums
   - Choice 2: Coke, Fanta, Sprite, Ginger Beer, Crème Soda
   - Choice 3: Doughnut, Chelsea Bun, Cheese Roll

   How many different party packs can be made?

2. Consider the word FLORIDA. You are required to form different seven-letter word arrangements using the letters of the word FLORIDA.
   How many possible word arrangements can be made if:
   (a) the letters may be repeated?
   (b) the letters may not be repeated?

3. Consider the word RANDOM. You are required to form different six-letter word arrangements using the letters of the word RANDOM.
   How many possible word arrangements can be made if:
   (a) the letters may be repeated?
   (b) the letters may not be repeated?

4. How many different ways are there of predicting the results of six soccer matches where each match can end in either a win or a lose?

5. A password is to be made up using the format XXXXY where X represents any digit from 0 to 9 and Y represents any letter of the alphabet. How many different passwords can be formed in each of the following cases?
   (a) The digits may be repeated as well as the letters of the alphabet.
   (b) The digits may not be repeated including the letters of the alphabet.
   (c) The digits and letters may be repeated but the number 0 and the vowels must be excluded.
   (d) The digits and letters may not be repeated and the number 0 and the vowels must be excluded.
   (e) The digits may be repeated but must be prime numbers and the letters may be repeated excluding the first five letters and the last five letters.

6. There are four bus lines between town A and town B and three bus lines between town B and town C.
   (a) In how many ways can a person travel by bus from town A to town C by way of town B?
   (b) In how many ways can a person travel return trip by bus from town A to town C by way of B?
   (c) In how many ways can a person travel return trip by bus from town A to town C by way of town B, if this person doesn’t want to use a bus line more than once?

Factorial notation

The product 6 × 5 × 4 × 3 × 2 × 1 can be written as 6! and this is read as “six factorial”.

In other words,
5! = 5 × 4 × 3 × 2 × 1
9! = 9 × 8 × 7 × 6 × 5 × 4 × 3 × 2 × 1
n! = n(n - 1)(n - 2)(n - 3)................3.2.1
Also 0! = 1
EXAMPLE 5
Consider the word PARKTOWN. You are required to form different eight-letter word arrangements using the letters of the word PARKTOWN. How many possible word arrangements can be made if the letters may not be repeated?

Solution
From Example 4, the solution to this question is \(8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320\) word arrangements.
We can write this in factorial notation:
\(8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 8!\)

EXAMPLE 6
In how many ways can 6 different people be seated in the first six seats in a movie theatre?

Solution
In this example, any one of the 6 people can be chosen to sit in the first seat. Thereafter, only one of the remaining 5 people can be chosen to sit in the second seat. After this, only one of the remaining 4 people can be chosen to sit in the third seat. After this, only one of the remaining 3 people can be chosen to sit in the fourth seat. After this, only one of the remaining 2 people can be chosen to sit in the fifth seat. The remaining person will then have to occupy the sixth seat. Therefore the number of ways that we can seat these people is as follows:

\(6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720\) ways

RULE 2
The number of arrangements of \(n\) different things taken in \(n\) ways is: \(n!\)

EXAMPLE 7
In how many ways can 7 vacant places be filled by 10 different people?

Solution
10 different people can occupy 7 places in the following ways:
10 \(\times\) 9 \(\times\) 8 \(\times\) 7 \(\times\) 6 \(\times\) 5 \(\times\) 4 \(\) (only 7 places can be occupied)

We can write this as follows:
\(10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 = \frac{10!}{3 \times 2 \times 1} = \frac{10!}{3!} = \frac{10!}{(10-7)!}\)

An important conclusion can be made based on this result:

RULE 3
The number of arrangements of \(n\) different things taken \(r\) at a time is given by \(\frac{n!}{(n-r)!}\)
EXAMPLE 8

There are 12 different singers that are hoping to occupy the first three places in SA Idols. In how many different ways can the first three places be occupied?

Solution

In this example there are 12 people to be arranged in 3 different ways.

The number of possible arrangements will be:

\[
\frac{12!}{(12-3)!} = \frac{12!}{9!} = 1320
\]

EXAMPLE 9

Consider the word LOVERS.

(a) How many six-letter word arrangements can be made if the letters may be repeated?
(b) How many six-letter word arrangements can be made if the letters may not be repeated?
(c) How many four-letter word arrangements can be made if the letters may be repeated?
(d) How many four-letter word arrangements can be made if the letters may not be repeated?

Solutions

(a) When the letters may be repeated, we use exponential notation:

\[
6 \times 6 \times 6 \times 6 \times 6 \times 6 = 6^6 = 46,656
\]

(b) When letters may not be repeated, we use factorial notation:

\[
6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720
\]

(c) \[6 \times 6 \times 6 \times 6 = 6^4 = 1296\]

(d) \[6 \times 5 \times 4 \times 3 = \frac{6!}{(6-4)!} = 360\]

EXERCISE 2

1. (a) In how many ways can 8 vacant places be filled by 8 different people?
   (b) In how many ways can 5 vacant places be filled by 15 different people?
2. Find the number of ways that a judge can award first, second and third places in a competition with ten contestants.
3. A school needs to employ four teachers out of a potential twenty. In how many ways can this be done?
4. Consider the word ORANGES.
   (a) How many seven-letter word arrangements can be made if the letters may be repeated?
   (b) How many seven-letter word arrangements can be made if the letters may not be repeated?
   (c) How many four-letter word arrangements can be made if the letters may be repeated?
   (d) How many four-letter word arrangements can be made if the letters may not be repeated?

5. Consider the word WISKUNDE.
   (a) How many eight-letter word arrangements can be made if the letters may be repeated?
   (b) How many eight-letter word arrangements can be made if the letters may not be repeated?
   (c) How many six-letter word arrangements can be made if the letters may be repeated?
   (d) How many six-letter word arrangements can be made if the letters may not be repeated?

6. The digits 0, 1, 2, 3, 4, 5, 6, 7 and 8 are used to make 4 digit codes.
   (a) How many different codes are possible if the digits may be repeated?
   (b) How many different codes are possible if the digits may not be repeated?
   (c) How many codes are numbers that are greater than or equal to 4 000 and are exactly divisible by 2? The digits may be repeated.
   (d) How many codes are numbers that are greater than 4 000 and are exactly divisible by 2?

ARRANGEMENTS OF OBJECTS IN A ROW

EXAMPLE 10

(a) In how many ways can three boys and two girls sit in a row?
(b) In how many ways can they sit in a row if a boy and his girlfriend must sit together?
(c) In how many ways can they sit in a row if the boys and girls are each to sit together?
(d) In how many ways can they sit in a row if just the girls are to sit together?
(e) In how many ways can they sit in a row if just the boys are to sit together?
(f) In how many ways can they sit in a row if the boys and girls are to alternate?

Solutions

(a) The five persons can sit in a row in \(5 \times 4 \times 3 \times 2 \times 1 = 5! = 120\) ways.
(b) The boy can sit to the left or right of his girlfriend: BG or GB
   Either the boy or girl could sit in the left seat. If it is the boy, then the girl will sit in the right seat. If the girl sits in the left seat, then the boy will sit in the right seat. From the counting principle, we can write this as \(2 \times 1 = 2!\)
   ways of sitting.
Now let’s treat the couple (BG or GB) as a single entity. This means that we have four “objects”: one couple, two boys and one girl. These four “objects” can be seated in 4! ways:

<table>
<thead>
<tr>
<th>couple (4 choices)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>couple (3 choices)</td>
<td></td>
</tr>
<tr>
<td>couple (2 choices)</td>
<td></td>
</tr>
<tr>
<td>couple (1 choice)</td>
<td></td>
</tr>
</tbody>
</table>

From the counting principle, the final answer is:

ways that the couple can sit together \times \text{ways that the couple sits with the other 3} \\
= 2! \times 4! = 48 \text{ ways}

**Alternative method:**

The couple can sit together in 2! ways.
There are 4 seating arrangements for the couple (shaded blocks):

<table>
<thead>
<tr>
<th>Seat 1</th>
<th>Seat 2</th>
<th>Seat 3</th>
<th>Seat 4</th>
<th>Seat 5</th>
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</tbody>
</table>

The other two boys and one girl can sit in the other seats in 3 \times 2 \times 1 = 3! ways.

Let’s put all of this together:
There are 4 possible seating arrangements for the couple and the others (see the table above).

There are 2! ways for the couple to sit together: BG or GB
The others can sit in 3 \times 2 \times 1 = 3! ways.
The answer is therefore: 4 \times 2! \times 3! = 48 \text{ ways}.

Notice that 4 \times 2! \times 3! = 4 \times 3 \times 2! = 4 \times 3 \times 2 \times 2! = 4! \times 2! (same as previous method’s answer).

(c) There are two ways to arrange them: BBBGG or GGBBB. In each case, the boys can sit in 3! ways and the girls can sit in 2! ways. Therefore, they can together sit in 2 \times 3! \times 2! = 24 \text{ ways}.
(d) There are 4 ways of arranging them: GGBBB or BGGB or BBGGB or BBBGG.
There are 2! ways for the girls to sit together.
The girls as an entity can sit with the other 3 boys in 4! ways.

| Two girls | Two girls | Two girls | Two girls |

The final answer is therefore: $2! \times 4! = 48$ ways

Alternatively:
In each case, the boys can sit in 3! ways and the girls can sit in 2! ways.
Therefore there are $4 \times 3! \times 2! = 48$ ways.

(e) There are 3 ways of distributing them: GGBBB or GBBBG or BBBGG.
There are 3! ways for the boys to sit together.
The boys as an entity can sit with the girls in 3! different ways.

| Three boys | Three boys | Three boys |

The final answer is therefore: $3! \times 3! = 36$ ways

Alternatively:
In each case, the boys can sit in 3! ways and the girls can sit in 2! ways.
Therefore there are $3 \times 3! \times 2! = 36$ ways.

(f) There is only one way that this can happen in this case: BGBGB (not GBGBB). Therefore there are $1 \times 3! \times 2! = 12$ ways.

EXERCISE 3

1. (a) In how many ways can nine people be seated in a row of nine chairs?
   (b) In how many ways can twenty people be seated in a row of twenty chairs?

2. (a) Five people are to be seated in a row. In how many ways can they be seated?
   (b) How many ways are there if two of the five insist on sitting next to each other in the row?
3. (a) In how many ways can five boys and four girls sit in a row?  
(b) In how many ways can they sit in a row if a boy and his girlfriend must sit together?  
(c) In how many ways can they sit in a row if the boys and girls are each to sit together?  
(d) In how many ways can they sit in a row if just the girls are to sit together?  
(e) In how many ways can they sit in a row if just the boys are to sit together?  
(f) In how many ways can they sit in a row if the boys and girls are to alternate?  

4. Four History books and three Geography books must be placed on a shelf.  
(a) In how many different ways can you arrange the books on the shelf?  
(b) If all the History books must be placed next to each other and all the Geography books must be placed next to each other, in how many ways can you arrange the books on the shelf?  
(c) If just the History books are to be together, in how many ways can you arrange the books on the shelf?  

5. In how many ways can four Mathematics books, three History books, three Science books and two Biology books:  
(a) be arranged on a shelf?  
(b) be arranged on a shelf so that all books of the same subject are together?  

6. Find the number of ways in which six people can ride a toboggan if one of a selected three must drive?  

7. In how many ways can three South Africans, four Americans, four Italians and two British citizens be arranged so that those of the same nationality sit together if they sit in a row?  

8. Golf balls are numbered 1 to 12 and placed in a box. Every time a ball is drawn it is placed on a rack, one next to the other.  
(a) How many different arrangements of the 12 golf balls is possible?  
(b) The numbers 9 and 12 must be placed next to each other in any order. In how many ways can the numbers then be re-arranged?  
(c) The numbers 1, 4, 5 and 8 are taken from the box. Two-digit numbers must be formed out of the four numbers. How many number arrangements can be formed if:  
(1) the digits may be repeated?  
(2) the digits may not be repeated?
LETTER ARRANGEMENTS WHERE LETTERS ARE REPEATED

EXAMPLE 11
Consider the letters of the word DAD.

(a) How many word arrangements can be made with this word if the repeated letters are treated as different letters?

(b) How many word arrangements can be made with this word if the repeated letters are treated as identical?

Solutions

(a) Let the first D be $D_1$ and the last D be $D_2$.

The word arrangements are:

$D_1 A D_2$ $D_2 A D_1$ $A D_1 D_2$

$A D_2 D_1$ $D_1 D_2 A$ $D_2 D_1 A$

There are 3 letters in the word DAD. Therefore there are $3! = 3 \times 2 \times 1 = 6$
possible word arrangements if the two D’s are treated as different letters. This is an application of Rule 2.

(b) Since the D’s are identical, we can remove the subscripts and obtain the following word arrangements:

DAD ADD DDA

There are 3 letters in the word DAD. Therefore there are

$3 = \frac{3 \times 2 \times 1}{2 \times 1} = \frac{3!}{2!}$ possible word arrangements if the two D’s are treated as identical letters.

In the number $\frac{3!}{2!}$, the numerator represents the total number of letters in the word DAD, which is 3. The denominator represents the fact that the letter D is written twice in the word.

RULE 4
The number of different ways that $n$ letters can be arranged where $m_1$ of the letters are identical, $m_2$ of the letters are identical, $m_3$ of the letters are identical, ……., $m_n$ of the letters are identical is given by:

$$\frac{n!}{m_1! \times m_2! \times m_3! \times \ldots \times m_n!}$$

where the repeated letters are treated as identical.
EXAMPLE 12

Consider the letters of the word NEEDED.

(a) How many word arrangements can be made with this word if the repeated letters are treated as different letters?

(b) How many word arrangements can be made with this word if the repeated letters are treated as identical?

(c) How many word arrangements can be made with this word if the word starts and ends with the same letter?

Solutions

(a) There are 6 letters in the word NEEDED. The total possible word arrangements (repeated letters are treated as different) is:

\[ 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720 \quad \text{(Rule 2)} \]

(b) The total possible word arrangements (repeated letters are treated as identical) is:

\[
\frac{6!}{3! \times 2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} = 60 \quad \text{(Rule 4)}
\]

The numerator represents the 6 letters in the word NEEDED. The denominator represents the three E’s and the two D’s.

(c) The only possibilities with the word NEEDED if you start and end with the same letter are:

<table>
<thead>
<tr>
<th>Option 1</th>
<th>Option 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>E</td>
</tr>
<tr>
<td></td>
<td>D</td>
</tr>
<tr>
<td></td>
<td>E</td>
</tr>
</tbody>
</table>

With the first option the letters in between the two D’s will be NEEE. The possible word arrangements will then be:

\[
\frac{4!}{3!} = \frac{4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 4 \quad \text{(Rule 4)}
\]

The numerator represents the 4 letters in the word NEEE (ignore the two D’s). The denominator represents the three E’s.

With the second option E the letters in between the two E’s will be NDED. The possible word arrangements will then be:

\[
\frac{4!}{2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1} = 12 \quad \text{(Rule 4)}
\]
The numerator represents the 4 letters in the word NDED (ignore the two E’s). The denominator represents the two D’s.

Therefore the total number of possible word arrangements that can be made if the word starts and ends with the same letter will be:

\[
\frac{4!}{3!} + \frac{4!}{2!} = 4 + 12 = 16
\]

**EXAMPLE 13**

Consider the letters of the word KNIGHT.

(a) How many word arrangements can be made with this word (letters are not repeated)?

(b) How many word arrangements can be made with this word if the word starts with K and ends with the T?

**Solutions**

(a) There are six different letters to be arranged in six positions.

The number of word arrangements are:

\[6! = 720\]

(b) The letters in between the K and T are NIGH. The possible word arrangements will then be:

\[
\frac{4!}{0!} = \frac{4 \times 3 \times 2 \times 1}{1} = 24
\]

(Rule 4)

The numerator represents the 4 letters in the word NIGH. The denominator represents the fact that no letters are repeated.

**EXERCISE 4**

1. Consider the word WINNERS.
   (a) How many word arrangements can be made with this word if the repeated letters are treated as different letters?
   (b) How many word arrangements can be made with this word if the repeated letters are treated as identical?
   (c) How many word arrangements can be made with this word if the word starts and ends with the same letter?
   (d) How many word arrangements can be made with this word if the word starts with W and ends with the S?
2. Consider the word WINNING.
   (a) How many word arrangements can be made with this word if the repeated letters are treated as different letters?
   (b) How many word arrangements can be made with this word if the repeated letters are treated as identical?
   (c) How many word arrangements can be made with this word if the word starts and ends with the same letter?
   (d) How many word arrangements can be made with this word if the word starts with W and ends with the G?
   (e) How many word arrangements can be made with this word if the word starts with the letter I?
   (f) How many word arrangements can be made with this word if the word ends with the letter N?

3. Consider the word TECHNOLOGY.
   (a) How many word arrangements can be made with this word if the repeated letters are treated as different letters?
   (b) How many word arrangements can be made with this word if the repeated letters are treated as identical?
   (c) How many word arrangements can be made with this word if the word starts and ends with the same letter?
   (d) How many word arrangements can be made with this word if the word starts with the letter O?
   (e) How many word arrangements can be made with this word if the word ends with the letter N?

4. Three Mathematics books and five Science books are to be arranged on a shelf.
   (a) In how many ways can these books be arranged if they are treated as separate books?
   (b) In how many ways can these books be arranged if they are treated as identical books?

5. There are six pool balls on a pool table. Some are red and some are blue. The red balls are identical to each other as well as the blue balls. The balls are removed from the table, one by one. How many different results can happen if there are:
   (a) five red balls?
   (b) four blue balls?
   (c) three of each colour?
EXAMPLES INVOLVING PROBABILITY

EXAMPLE 14

Consider the letters of the word DREAMS. If the letters are arranged in any order without repetition to form different words, what is the probability that the word formed will start with D and end with S?

Solution

The number of ways that the six letters can be arranged is 6!

Let event E be defined as the event that the word formed will start with D and end with S.

The number of arrangements for event E is $\frac{4!}{4!} = 1$

Therefore the probability of event E happening is:

$$\frac{4!}{6!} = \frac{4!}{6 \times 5 \times 4!} = \frac{1}{30}$$

EXAMPLE 15

A combination to a lock is formed using three letters of the alphabet, excluding the letters O, Q, S, U, V and W and using any three digits. The numbers and letters can be repeated. Calculate the probability that a combination, chosen at random:

(a) starts with the letter X and ends with the number 6.
(b) has exactly one X.
(c) has one or more number 6 in it.

Solutions

(a) Let A be the event that a number plate starts with the letter X and ends with the number 6.

Since 20 letters and 10 digits can be used, the number of plates possible will be:

$$20 \times 20 \times 20 \times 10 \times 10 \times 10 = 8000000$$

For event A, the number of possibilities is reduced to:

$$1 \times 20 \times 20 \times 10 \times 10 \times 10 = 40000$$

Therefore, the probability of event A happening is: $\frac{40000}{8000000} = \frac{1}{200}$
(b) Let $B$ be the event of choosing exactly one $X$.

The number of possible ways of event $B$ happening is:

$$(1 \times 19 \times 19 \times 10 \times 10 \times 10) + (19 \times 1 \times 19 \times 10 \times 10 \times 10) + (19 \times 19 \times 1 \times 10 \times 10 \times 10)$$

$= 1 083 000$

Therefore, the probability of event $B$ happening is:

$$\frac{1 083 000}{8 000 000} = \frac{1 083}{8 000}$$

(c) Let $C$ be the event of at least one 6 being chosen.

**Method 1**

Total number of possible combinations:

$20 \times 20 \times 20 \times 10 \times 10 \times 10 = 8 000 000$

Number of combinations without 6:

$20 \times 20 \times 20 \times 9 \times 9 \times 9 = 5 832 000$

Number of combinations with at least one 6:

$20 \times 20 \times 20 \times 10 \times 10 \times 10 - 20 \times 20 \times 20 \times 9 \times 9 \times 9$

$= 8 000 000 - 5 832 000$

$= 2 168 000$

The probability of at least one 6 is:

$$\frac{2 168 000}{8 000 000} = \frac{217}{1 000}$$

**Method 2**

The probability of event $C$ happening can determined by using the fact that

$$P(C) = 1 - P(\text{not } C).$$

$$\therefore P(C) = 1 - \frac{20 \times 20 \times 20 \times 9 \times 9 \times 9}{8 000 000}$$

$$\therefore P(C) = 1 - \frac{729}{1 000}$$

$$\therefore P(C) = \frac{271}{1 000}$$
EXAMPLE 16

Consider the letters of the word NEEDED. What is the probability that the word arrangement formed will start and end with the same letter? The repeated letters are identical.

Solution

The only possibilities with the word NEEDED if you start and end with the same letter are:

<table>
<thead>
<tr>
<th>Option 1</th>
<th>Option 2</th>
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</thead>
<tbody>
<tr>
<td>D</td>
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<tr>
<td>D</td>
<td>E</td>
</tr>
</tbody>
</table>

With the first option the letters in between the two D’s will be NEEE. The possible word arrangements will then be:

\[
\frac{4!}{3!} = \frac{4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 4
\]

(Rule 4)

The numerator represents the 4 letters in the word NEEE (ignore the two D’s). The denominator represents the three E’s.

With the second option E the letters in between the two E’s will be NDED.

The possible word arrangements will then be:

\[
\frac{4!}{2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1} = 12
\]

Therefore the total number of possible word arrangements that can be made if the word starts and ends with the same letter will be:

\[
\frac{4!}{3!} + \frac{4!}{2!} = 4 + 12 = 16
\]

However, the sample space in this example (the total possible word arrangements where repeated letters are treated as identical) is:

\[
\frac{6!}{3! \times 2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} = 60
\]

Therefore the probability that the word arrangement formed will start and end with the same letter is:

\[
\frac{4!}{3! \times 2!} \times \frac{4!}{6!} = \frac{4}{15}
\]
EXERCISE 5

1. Consider the letters of the word KNIGHT. If the letters are arranged in any order without repetition to form different words, what is the probability that the word formed will:
   (a) start with K and end with T?
   (b) start with the letter N?

2. Consider the letters of the word CLIPBOARD. If the letters are arranged in any order without repetition to form different words, what is the probability that the word formed will:
   (a) start with C and end with D?
   (b) start with the letter O?

3. A password is formed using three letters of the alphabet, excluding the letters A, E, I, O and U and using any three digits, excluding 0. The numbers and letters can be repeated. Calculate the probability that a password, chosen at random:
   (a) starts with the letter B and ends with the number 4.
   (b) has exactly one B.
   (c) has at least one 4.

4. Determine the probability of getting a ten digit cell-phone number if the first digit is even, none of the first three must be 0 and none of the digits may be repeated.

5. Consider the letters of the word WINNERS. The repeated letters are identical. If the letters are arranged in any order without repetition to form different words, what is the probability that the word formed will:
   (a) start with W and end with S?
   (b) start with the letter N?

6. Consider the letters of the word WINNING. The repeated letters are identical. If the letters are arranged in any order without repetition to form different words, what is the probability that the word formed will:
   (a) start with W and end with G?
   (b) start with the letter N?

7. Consider the letters of the word MATHEMATICIAN. The repeated letters are identical. If the letters are arranged in any order without repetition to form different words, what is the probability that the word formed will:
   (a) start and end with the same letter?
   (b) end with the letter N?

8. Four Mathematics books, three History books, three Science books and two Biology books are arranged randomly on a shelf. What is the probability that:
   (a) all books of the same subject land up next to each other?
   (b) just the History books will be together?

9. Seven boys and six girls are to be seated randomly in a row. What is the probability that:
   (a) the row has a boy at each end?
   (b) the row has boys and girls sitting in alternate positions?
   (c) two particular girls land up sitting next to each other?
   (d) all the girls sit next to each other?
REVISION EXERCISE

1. In a company there are three vacancies. The company had identified candidates to fill each post.

<table>
<thead>
<tr>
<th>POST</th>
<th>CANDIDATES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Floor worker</td>
<td>Carl, James and Sally</td>
</tr>
<tr>
<td>Sales representative</td>
<td>Thandi, Sandy, Sizwe and David</td>
</tr>
<tr>
<td>Sales manager</td>
<td>Mpho and Nicole</td>
</tr>
</tbody>
</table>

(a) In how many different ways can these three posts be filled?
(b) If it is certain that James will get the job as clerk, in how many different ways can the three posts be filled?

2. You have a combination lock with four numbers on your locker at school. If someone tries to get into your locker to steal your cell-phone, what is the probability that he will get the combination correct on his first attempt?

3. Consider the word CERTAINLY.
   (a) How many nine-letter word arrangements can be made if the letters may be repeated?
   (b) How many nine-letter word arrangements can be made if the letters may not be repeated?
   (c) How many four-letter word arrangements can be made if the letters may be repeated?
   (d) How many four-letter word arrangements can be made if the letters may not be repeated?
   (e) How many word arrangements can be made with this word if the word starts with C and ends with Y?
   (f) What is the probability that the word starts with C and ends with Y?

4. Consider the word ADVERTISEMENT.
   (a) How many word arrangements can be made with this word if the repeated letters are treated as different letters?
   (b) How many word arrangements can be made with this word if the repeated letters are treated as identical?
   (c) How many word arrangements can be made with this word if the word starts and ends with the same letter?
   (d) What is the probability that the word starts and ends with the same letter?
   (e) How many word arrangements can be made with this word if the word starts with the letter M?
   (f) What is the probability that the word ends with the letter M?
5. The Gauteng government introduced a new licence-plate system in December 2010, the month it expected to run out of numbers to allocate under the old system. The number plates for motor vehicles in Gauteng used to be created as follows:

Three letters excluding the vowels A, E, I, O, U as well as Q (repetition allowed)

For example:

| Y | G | G | 8 | 6 | 6 |

The latest number plates for Gauteng have been changed. The same letters and digits are used as before but the structure is as follows:

Two letters Two digits Two letters

For example:

| B | B | 0 | 1 | C | Y |

(a) How many possible number plates were there with the old system?
(b) How many number plates are there with the new system?
(c) Why do you think that the Traffic Department changed to the new system?

6. The digits 0, 1, 2, 3, 4, 5, 6 and 7 are used to make 4 digit codes.

(a) How many unique codes are possible if the digits can be repeated?
(b) How many unique codes are possible if the digits cannot be repeated?
(c) In the case where digits may be repeated, how many codes are numbers that are greater than 2 000 and even?
(d) In the case where digits may not be repeated, how many codes are numbers that are greater than 2 000 and divisible by 4?
(e) What is the probability that a code will contain at least one 7? The digits may be repeated.
(f) What is the probability that a code will contain at least one 7? The digits may not be repeated.
(g) How many codes can be formed between 4 000 and 5 000? The digits may be repeated.

7. There are 7 different shirts and 4 different pairs of trousers in a cupboard. The clothes have to be hung on the rail.

(a) In how many different ways can the clothes be arranged on the rail?
(b) In how many different ways can the clothes be arranged if all the shirts are to be hung next to each another and the pairs of trousers are to be hung next to each another on the rail?
(c) In how many ways can a pair of trousers hang at the beginning of the rail and a shirt will hang at the end of the rail?
(d) What is the probability that a pair of trousers will hang at the beginning of the rail and a shirt will hang at the end of the rail?
(e) A chosen shirt and trouser must be hung together. In how many different ways can this be done?
SOME CHALLENGES

1. In a survey, 1470 vegetarians were asked if they eat eggs. The results of the survey were as follows:

<table>
<thead>
<tr>
<th></th>
<th>Eat eggs</th>
<th>Don’t eat eggs</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>433</td>
<td>b</td>
<td>752</td>
</tr>
<tr>
<td>Female</td>
<td>a</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>Total</td>
<td>883</td>
<td>587</td>
<td>1470</td>
</tr>
</tbody>
</table>

(a) Determine the values of $a$, $b$, $c$ and $d$.
(b) Determine the probability of choosing at random a vegetarian who does eat eggs.
(c) Determine the probability of choosing at random a female vegetarian who doesn’t eat eggs.
(d) A female is chosen. Determine the probability of choosing at random a female vegetarian who does eat eggs.
(e) Is being a female vegetarian and not eating eggs independent? Give a reason. Use calculations and round off your answer to one decimal place.

2. A coin is weighted so that $P(H) = \frac{2}{3}$ and $P(T) = \frac{1}{3}$ is tossed.
If heads appears, then a number is selected at random from the numbers 1 to 9. If tails appears, then a number is selected at random from the numbers 1 to 5.
Draw a tree diagram and determine the probability that an even number will be selected.

3. A bag contains 3 red marbles and 7 white marbles. A marble is drawn from the bag and a marble of the other colour is then put into the bag. A second marble is drawn from the bag.
(a) Draw a tree diagram.
(b) Determine the probability that the second marble drawn is red.

4. Calculate the number of ways in which 3 girls and 4 boys can be seated in a row of 7 chairs if each arrangement is to be symmetrical.

5. A card is chosen at random from a pack of cards. It is then replaced. This experiment is carried out 520 times. Determine the expected number of times for which the card is:
(a) a club (b) an ace (c) either an ace or club

6. You have seven fruits to last the week. Two are bananas, three are apples and two are peaches. You select one fruit at random each day. In how many ways can you select the fruits?
CHAPTER 1
REVISION EXERCISE
1(a) \( T_n = 2n^2 - 2n + 2 \) 1(b) \( T_n = n^2 + 2n + 1 \) 1(c) \( T_n = \frac{1}{2} n^2 + \frac{1}{2} n \) 1(d) \( T_n = n^2 - 2n \)
1(e) \( T_n = -n^2 - 2 \) 1(f) \( T_n = \frac{1}{2} n^2 - \frac{11}{2} n + 15 \) 2(a) \( T_n = 2n^2 - n \) and \( T_{20} = 780 \)
2(b) \( n = 40 \) 3(a) \( T_n = -n^2 - 3n \) and \( T_{25} = -700 \) 3(b) \( n = 30 \) 4(a) 55, 78
4(b) Quadratic 4(c) \( T_n = 2n^2 + n \)
EXERCISE 1
1(a) \( T_n = 4n - 5 \) 1(b) \( T_n = -6n + 10 \) 1(c) \( T_n = -2n + 3 \) 1(d) \( T_n = 7n + 92 \) 2(a) \( T_{38} = -152 \)
2(b) \( T_{38} = -127.5 \) 2(c) \( T_{38} = -308 \) 2(d) \( T_{38} = -\frac{87}{4} \) 2(e) \( T_{38} = 110 \) 2(f) \( T_{38} = -71 \)
3(a) \( n = 34 \) 3(b) \( n = 48 \) 3(c) \( n = 44 \) 3(d) \( n = 21 \) 4(a) \( T_{31} = 151 \)
4(b) \( n = 11 \) 5(a) \( n = 17 \) 5(b) \( T_{31} = 187 \) 6(a) \( p = \frac{1}{2} \) 6(b) \( \frac{1}{2} ; 3; 5 \frac{1}{2} \)
6(c) \( T_{49} = 120 \frac{1}{2} \) 6(d) \( n = 41 \) 7(a) \( T_{10} = 10x + 30 \) 7(b) \( T_n = nx + 3n \)
EXERCISE 2
1(a) \( T_n = 2\left(-\frac{1}{2}\right)^{n-1} \) 1(b) \( T_n = 2(4)^{n-1} \) 1(c) \( T_n = -\frac{1}{2}(3)^{n-1} \) 1(d) \( T_n = (0,2)^{n-1} \) 2(a) \( T_9 = \frac{1}{2} \)
2(b) \( T_9 = 64 \) 2(c) \( T_9 = 2916 \) 2(d) \( T_9 = 17 \frac{25}{33} \) 2(e) \( T_9 = 10 \frac{7}{8} \) 2(f) \( T_9 = 9 \frac{5}{8} \)
3(a) \( n = 8 \) 3(b) \( n = 10 \) 3(c) \( n = 10 \) 4(a) \( T_{10} = 320 \) 4(b) \( n = 8 \)
5(c) \( n = 10 \) 5(d) \( n = 16 \) 6. \( 4; 10; 25 \) 7(a) \( t = -\frac{1}{3} \)
7(b) \( \frac{1}{3} ; \frac{4}{3} ; \frac{8}{9} \) 7(c) \( T_{10} = \frac{1024}{3} \) 7(d) \( n = 5 \)
EXERCISE 3
1(a) \( 20 \) 1(b) \( 116 \) 1(c) \( 21 \) 1(d) \( 31 \frac{1}{2} \) 1(e) \( 32 \) 1(f) \( 139 \) 2(a) \( \sum_{k=1}^{6} 2k \)
2(b) \( \sum_{k=1}^{5} k^3 \) 2(c) \( \sum_{k=1}^{3} \frac{1}{k^2} \) 2(d) \( \sum_{k=1}^{9} 3(-2)^{k-1} \) 2(e) \( \sum_{k=1}^{24} \frac{k}{k+1} \) 2(f) \( \sum_{k=1}^{6} 5 \)
2(g) \( \sum_{k=1}^{7} \frac{1}{k} \)
EXERCISE 4
1(a) \( S_{20} = 670 \) 1(b) \( S_{32} = -1168 \) 1(c) \( S_{100} = 23350 \) 1(e) \( S_{100} = 15350 \) 2(a) \( S_{100} = 15350 \)
2(b) \( S_{49} = -2303 \) 2(c) \( S_{50} = 4850 \) 3(a) \( n = 80 \) 3(b) \( S_80 = 12400 \) 4(a) \( n = 22 \)
4(b) \( S_{22} = 357.5 \) 5(a) \( S_{12} = -654 \) 5(b) \( = 16284 \) 5(c) \( S_{18} = -837 \) 6. \( S_{50} = 250000 \)
7(a) \( n = 20 \) 7(b) \( n = 22 \) 7(c) \( n = 11 \) 8(a) \( m = 18 \) 8(b) \( m = 35 \) 8(c) \( m = 21 \) 9. \( n = 9 \)
10. \( \frac{135}{202} \) 11(a) \( 10 \) 11(b) \( 50 \) 11(c) \( n \) 12(a) \( T_{15} = 48000 \)
12(b) \( S_{15} = 40500 \) 12(c) 139 days 13(a) Shaun: 54, Mpho: 56 13(b) 8 weeks
13(c) \( 18225 \) 14. \( 34687.5 \)
EXERCISE 5
1(a) \( S_9 = 728.96 \) 1(b) \( S_{10} = -42,625 \) 1(c) \( S_{15} = 9863.85 \) 2. \( S_{12} = -341,25 \) 3(a) \( n = 12 \)
3(b) \( -127.97 \) 4. \( S_6 = -24 \frac{17}{27} \) 5(a) \( S_{10} = 48828.12 \) 5(b) \( S_9 = 31,9375 \) 5(c) \( S_{11} = 121,5 \)
6(a) \( n = 8 \) 6(b) \( n = 6 \) 6(c) \( n = 8 \) 7(a) \( m = 9 \) 7(b) \( m = 6 \) 7(c) \( m = 100 \)
8. \( p = 10 \) 9(a) R131072 9(b) R262143
EXERCISE 6
1(a) \( 5; 14; 23 \) 1(b) \( 3; 7; 11 \) 2. \( T_{100} = 610 \) 3(a) \( n = 19 \) 3(b) \( n = 4 \frac{1}{2} \)
4(a) \( a = -26, d = \frac{11}{4} \) 4(b) \( S_{21} = 189 \) 4(c) \( T_{12} = 248 \) 5. \( n = 29 \) 6. \( a = 1, d = 3 \)
7(b) \( 429429 \) 8(a) \( \frac{7}{8} ; \frac{7}{4} ; \frac{7}{2} \) 8(b) \( 2; 3; \frac{9}{2} \) or \( -2; 3; -\frac{9}{2} \) 9(a) \( a = \frac{5}{6} \) 9(b) \( n = 15 \)
328
10. \( n = 10 \)  
11a(1) \(-\frac{3}{2}\)  
11a(2) \(S_{10} = -725,3\)  
11b) \(13,28125\)  
12. \( r = 2 \)

**EXERCISE 7**

1(a) \(S_n = 3\)  
1(b) \(S_n = -42\frac{2}{3}\)  
1(c) \(S_n = 20\frac{3}{7}\)  
2(a) \(S_n = 2\frac{3}{2}\)

2(b) \(S_n = 2\)  
2(c) \(S_n = \frac{10}{9}\)  
3(a) \(\frac{23}{99}\)  
3(b) \(\frac{4\frac{2}{9}}{9}\)  
3(c) \(\frac{49}{90}\)  
4(a) \(-1 < x < 0\)

4(b) \(1 < x < 3\)  
4(c) \(2 < x < 4\)  
5(a) \(r = -\frac{15}{16}\)  
5(b) \(15,6\)  
6. \(1; -\frac{2}{3}; \frac{4}{9}\)

**REVISION EXERCISE**

1(a) \(\sum_{k=1}^{31}(3k-2)\)  
1(b) \(\sum_{k=1}^{57}(-2k+3)\)  
1(c) \(\sum_{k=1}^{\infty}(\frac{1}{2})^{k-1}\)  
1(d) \(\sum_{k=1}^{\infty}9^{k-1}\)  
2. \(S_2 = 2\frac{61}{233}\)

3(a) \(S_{50} = 2500\)  
3(b) \(S_n = 3,5\)  
4. \(n = 20\)  
5. \(T_{60} = 3720\)

6(a) Geometric

7. \(x = 1, y = 0\) or \(x = 5, y = 8\)

8(a) \(T_2 = 8,5\) and \(T_3 = 5\)

8(b) \(T_2 = 6\) and \(T_3 = 3\)

9(c) \(S_{10} = 23\frac{165}{128}\)

9. \(T_{21} = 62\)

10. \(T_{49} = -172\)

11. \(S_{20} = 3,486,784,400\)

12(a) \(\frac{8}{5} ; 8 ; 40\)  
12(b) Term 4  
13. \(\frac{2}{729}\)  
14. \(0 < x < \frac{2}{3}\)

15(a) \(12,8\) m  
15(b) \(n = 7\)  
15(c) \(S_n = 20\) \(\frac{1}{0,8} = 100m\)  
16. \(x = 3\)

17. \(a = 12\) or \(a = 10\)  
18. \(r = \frac{1}{3}\)

19(a) \(\sqrt{2} \frac{1}{2}\)  
19(b) \(\frac{1}{2}\)  
19(c) \(2 m^2\)  
19(d) 13,666m

**SOME CHALLENGES**

1. \(r = \frac{1}{3}\)  
2. \(\sum_{k=0}^{\infty}x^{n-k}y^k\)  
3. \(p = 5\)

4(b) 72

5. \(x = 1\) and \(y = 1\)

9(a) \(T_3 = 5\)  
9(b1) \(T_8 = 46\)  
9(b2) \(n = 10\)

12(a) \(n = k\)  
12(b) \(\sum_{k=0}^{\infty}\frac{x}{2^n} = \frac{k}{4}\)  
12(c) \(168\frac{2}{3}\)

13. 351 cans

**CHAPTER 2**

**EXERCISE 1**

(a) Function; one-to-one; Domain: \(x \in [-3; 0; 2]\) ; Range: \(y \in [0; 3; 5]\)

(b) Function; one-to-one; Domain: \(x \in (-\infty ; \infty)\) ; Range: \(y \in (-\infty ; \infty)\)

(c) Function; many-to-one; Domain: \(x \in (-\infty ; \infty)\) ; Range: \(y \in [0\); 

(d) Function; many-to-one; Domain: \(x \in (-\infty ; \infty)\) ; Range: \(y \in [-4 ; \infty)\)

(e) Function; many-to-one; Domain: \(x \in (-\infty ; \infty)\) ; Range: \(y \in (-\infty ; -1]\)

(f) Not a function; Domain: \(x \in [0 ; \infty)\) ; Range: \(y \in (-\infty ; \infty)\)

(g) Function; many-to-one; Domain: \(x \in [-2 ; 2]\) ; Range: \(y \in [0 ; 2]\)

(h) Not a function; Domain: \(x \in [-2 ; 2]\) ; Range: \(y \in [-2 ; 2]\)

(i) Function; one-to-one; Domain: \(x \in (-\infty ; \infty)\) ; Range: \(y \in (0 ; \infty)\)

(j) Function; many-to-one; Domain: \(x \in (-\infty ; \infty)\) ; Range: \(y \in (-\infty ; \infty)\)

**EXERCISE 2**

(a)(1) \(f^{-1}(x) = \frac{x-4}{2}\)

(a)(2) \(\frac{x}{2}\)

(b)(1) \(f^{-1}(x) = \frac{x+6}{3}\)

(b)(2) \(\frac{x}{3}\)
(a)(4) \((-4; -4)\)
(c)(1) \(f^{-1}(x) = 2x - 6\)
(c)(2)

(b)(3) \((3; 3)\)
(d)(1) \(f^{-1}(x) = \frac{x + 1}{4}\)
(d)(2)

(c)(3) \((6; 6)\)
(d)(3) \(\left(\frac{1}{3}; \frac{1}{3}\right)\)

(e)(1) \(f^{-1}(x) = -\frac{1}{2}x\)
(e)(2)

(f)(1) \(f^{-1}(x) = \frac{x + 5}{2}\)
(f)(2)

(e)(3) \((0; 0)\)
(f)(3) \((5; 5)\)

EXERCISE 3

1(a)(1) Situation 1: \(f(x) = 2x^2\) where \(x \geq 0\)
Situation 2: \(f(x) = 2x^2\) where \(x \leq 0\)
1(a)(2)

1(a)(3) Situation 1: \(f(x) = 2x^2\) where \(x \geq 0\)
Situation 2: \(f(x) = 2x^2\) where \(x \leq 0\)

1(a)(4) Situation 1: \(f^{-1}(x) = \sqrt{x}\) where \(x \geq 0\)
Situation 2: \(f^{-1}(x) = -\sqrt{x}\) where \(x \geq 0\)
1(a)(5)

1(b)(1)

1(b)(2) Situation 1: \(g(x) = 3x^2\) where \(x \geq 0\)
Situation 2: \(g(x) = 3x^2\) where \(x \leq 0\)
1(b)(3)

1(b)(4)

1(b)(5)
\[ g^{-1}(x) = \sqrt{x/3} \quad \text{where } x \geq 0 \]

\[ g^{-1}(x) = -\sqrt{x/3} \quad \text{where } x \geq 0 \]

**1(b)(5)**

**Situation 1**

- **Domain of** \( g \): \( x \in [0; \infty) \)
- **Range of** \( g : \) \( y \in [0; \infty) \)
- **Domain of** \( g^{-1} : \) \( x \in [0; \infty) \)
- **Range of** \( g^{-1} : \) \( y \in [0; \infty) \)

**Situation 2**

\[ \begin{align*}
1(c)(1) & : g(x) = x^2, \quad x \leq 0 \\
1(c)(2) & : f(x) = x^2, \quad x \geq 0 \\
1(d)(1) & : g(x) = -x^2, \quad x \leq 0 \\
1(d)(2) & : f(x) = -2x^2, \quad x \leq 0 \\
1(d)(3) & : g(x) = \frac{-x}{2}, \quad x \geq 0 \\
1(d)(4) & : f(x) = \frac{-x}{2}, \quad x \leq 0 \\
1(e)(1) & : g(x) = \frac{1}{2}x^2, \quad x \geq 0 \\
1(e)(2) & : g(x) = \frac{1}{2}x^2, \quad x \leq 0 \\
1(e)(3) & : g(x) = \frac{1}{2}x^2, \quad x \leq 0
\end{align*} \]
2(c)(1) \( y = \frac{1}{2} x^2 \) or \( y = x^{\frac{3}{2}} \) where \( x \neq 0 \)

1(f)(1) \( y = \frac{1}{2} x^2 \) or \( y = x^{\frac{3}{2}} \) where \( x \neq 0 \)

2(a) \( A(1;1) \)

2(b) \( f^{-1}(x) = x^2 \) where \( x \geq 0 \)

2(c)(1) \( f^{-1}(x) = x^2 \) for \( x \geq 0 \)

2(c)(2) \( y = -\frac{1}{x} \) for \( x < 0 \)

2(c)(3) \( y = -\sqrt{x} \) for \( x \geq 0 \)

EXERCISE 4

1(a)

1(b)

1(c)

1(d)

1(e)

2(f) Domain of \( f \): \( x \in (-\infty; 0) \)
Range of \( f \): \( y \in (0; \infty) \)
Domain of \( g \): \( x \in (-\infty; 0) \)
Range of \( g \): \( y \in (0; \infty) \)
EXERCISE 5
1(a) \( \log_8 3 + \log_8 \, x \) \quad 1(b) \( \log_8 6 - \log_8 \, x \) \quad 1(e) \( 4 \log_8 \, x \) \quad 1(d) \( \log_3 \, x + 2 \log_3 \, y \)
1(e) \( 2 \log_4 \, a + 5 \log_4 \, b \) \quad 1(f) \( \log_5 \, 2 + 2 \log_5 \, m - \log_5 \, n \) \quad 1(g) \( \log_3 \, 5 + 3 \log_2 \, a - 2 \log_2 \, b - \log_2 \, c \)
1(h) \( 2 \log_m \, (x + y) - \log_m \, x \) \quad 2(a) \( \log_4 \, (xy) \) \quad 2(b) \( \log_4 \, \frac{x}{y} \) \quad 2(c) \( \log_4 \, x^3 \)
2(d) \( \log_8 \, \frac{a^3}{c^5} \) \quad 2(e) \( \log_3 \, \frac{x(x + y)}{y} \) \quad 2(f) \( \log_2 \, \frac{3y^2}{x^5} \)

EXERCISE 6
(a) \( \frac{12}{9} \) \quad (b) \( \frac{5}{6} \) \quad (c) \( 2 \) \quad (d) \( -4 \)
(b) \( \frac{17}{6} \) \quad (i) \( 6 \) \quad (j) \( -2 \) \quad (k) \( 10 \) \quad (l) \( \frac{7}{3} \) \quad (m) \( \frac{1}{2} \) \quad (n) \( 12 \)
(o) \( -14 \) \quad (p) \( \frac{5}{2} \) \quad (q) \( 1 \) \quad (r) \( 0 \) \quad (s) \( \frac{1}{3} \) \quad (t) \( 2 \)

EXERCISE 7
(a) \( 1 \) \quad (b) \( 1 \) \quad (c) \( 3 \) \quad (d) \( -1 \) \quad (e) \( 2 \) \quad (f) \( 2 \) \quad (g) \( 3 \)

EXERCISE 8
1(a) \( x = 1,262 \) \quad 1(b) \( x = 0,580 \) \quad 1(e) \( x = 0,631 \) \quad 1(d) \( x = -3,322 \) \quad 1(e) \( x = 1,161 \)
1(f) \( x = 46,556 \) \quad 1(g) \( x = -0,096 \) \quad 1(h) \( x = 0,052 \) \quad 2(a) \( x = 8 \) \quad 2(b) \( x = 9 \)
2(c) \( x = 1 \) \quad 2(d) \( x = 10 \) \quad 2(e) \( x = 10,000 \) \quad 2(f) \( x = 5 \) \quad 2(g) \( x = -2 \)
2(h) \( x = 2 \) \quad 2(i) \( x = 4 \) \quad 2(j) \( x = 2 \) \quad 2(k) \( x = 5 \) \quad 2(l) \( x = 3 \)

EXERCISE 9
1(a) \( 900 \) \quad 1(b) \( 15 \) \quad 3 \quad m \quad 2(a) \( 2 \) \quad mg \quad 2(b) \( 30\% \) \quad 2(c) \( 2 \) \quad hours
3(a) \( 1039,3117,5400 \) \quad 3(b) \( 6 \) \quad hours \quad 4(a) \( 17100 \) \quad years
5(a) \[ \begin{array}{cccccccc}
\text{Time } t (\text{in days}) & 0 & 1 & 2 & 3 & 4 & 5 & n \\
\text{Area } y \text{ covered in square metres} & 1 & 2 & 4 & 8 & 16 & 32 & 2^n \\
\end{array} \]
5(b) \( A = 2^n \) \quad 5(c) \( A = 2^n \) \quad 5(d) \( 8^{th} \) day
REVISION EXERCISE

1(a) \( f(x) = 2x^2 \) where \( x \geq 0 \) or \( f(x) = 2x^2 \) where \( x \leq 0 \)

1(b) \[ y = x \]

2(a) \( a = 2 \) \( \rightarrow \) \( f^{-1}(x) = \log_2 x \)

2(d) \( x \in (0 ; \infty) \)

2(e) see diagram

3(a) \( a = 2 \) \( \rightarrow \) \( f^{-1}(x) = \log_2 x + 1 \)

3(c) \( \rightarrow \) \( f^{-1}(x) = x + 1 \)

3(d) \( x \in (0 ; \infty) \)

4(b) \( \text{The inverse is a one-to-many relation, which is not a function.} \)

4(c) \( x \in (0 ; \infty) \)

4(d) \( \text{The inverse is a one-to-many relation, which is not a function.} \)

4(e)(1) \( x \geq 0 \) or \( x \leq 0 \)

4(e)(2) \( x \in (0 ; \infty) \)

4(f)(1) Reflect about the line \( y = x \) and then the \( x \)-axis

4(f)(2) Reflect about the \( y \)-axis and then translate 1 unit upwards

5(b) \( y = 4^x \)

5(c) \( y = -\log_4 x \)

5(d) \( y = -\log_4 x \)

SOME CHALLENGES

1(a) No, the function is many to one.

1(b) \( [0 ; 2] \)

1(c) \( 325 \log 2 \)

1(d) One to many

2(a) 325 log 2 \( \rightarrow \) \( 2(b) 9863.85 \)
CHAPTER 3
REVISION EXERCISE
1(a) 115 129.86 1(b) 29.3% 2. 20 000 3(a) 0.16985856 3(b) 315 588.82
4. 207 504.74 5. 11 598.77
EXERCISE 1
1(a) 10 y 3 m 1(b) Approx 10 yrs 1(c) 10 y 1 m 1(d) Approx 10 yrs
1(e) 14 y 3 m 2(a) 6 y 8 m 2(b) 4 y 9 m 3. 5 y 11 m
4. Approx 2 yrs 5. 5 y 2 m 6. 1650 7. 7 y 11 m
8(a) 12.9% 8(b) Approx 5 yrs
EXERCISE 2
1. 1(a) 2. 1(b) 3 1(c) 9 1(d) 8 1(e) 7 1(f) 5 1(g) 241
1(h) 240 1(i) 238 1(j) 238 1(k) 236 1(l) n 1(m) n+1 1(n) n−1
2. 45 340.52 3. 333 298.22 4. 1740 210.05 5. 691 094.69 6. 27 931.16
7. 324 205.53 8. 4791.20 9. 2048.10 10. 610 658.55
EXERCISE 3
1(a) 88 792.08 1(b) 108 184.53 2(a) 314 781.65 2(b) 330 520.73 3. 62 807 510.92
4(a) 142 237.76 4(b) 138 227.76 5. 17 6. 18 7. 19
EXERCISE 4
1. 203 301.92 2. 276 874.66 3. 36 803.17 4. 5565.02 5. 2104.94
6. 1801.85 7(a) 131 534 7(b) 2954.38 8. 8581.05 9. 55 321.37
10. 12 435.21 11(a) 1354.67 11(b) 19 382.47
EXERCISE 5
1. 1198.33 2. 2667.29 3. 46 538.67 4. 16 911.34 5. 19 396.79
6(a) 35 : 3980.23 6(b) 36 months 7. 73 quarters or 19 years 3 months 8. No
EXERCISE 6
EXERCISE 7
1(a) 131 534 1(b) 107 768.23 1(c) 32 722.52 2(a) 36 803.17 2(b) 15 825.44
3(a) 3160.06 3(b) 76 841.42
EXERCISE 8
1(a) 36 700.16 1(b) 377 937.58 1(c) 341 237.42 1(d) 3132.50 1(e)(1) 34 166.75
1(e)(2) 3446.14 2(a) 61 674.35 2(b) 499 551.48 2(c) 5661.61
REVISION EXERCISE
1(a) 100 000 1(b) 25 000 1(c) reducing-balance 1(d) 29.3% 2. 11 y 6 m
3. Approx 20 yrs 4(a) 104 953.38 4(b) 109 222.11 5. 9280.85
6. 1 001 133.53 7. 19 8. 225 418.97 9(a) 650 000 9(b) 10 467.74
9(c) 597 115.31 10(a) 0.1236 10(b) 36 856.60 11(a) 1730.21 11(b) Option 2
11(c) 11.52% 12(a) 263 356.03 12(b) 4685.48 13. 4 14(a) 133 929.25
14(b) 3636.36 14(c) 190 602.72 14(d) 5760.72
SOME CHALLENGES
1. 37 259.14 2. 15 282.91 3. 49 678.81 4. 13 093.44 5(a) 6934.45
5(b) 174 122.75 6(a) 800 350.21 6(b) 10 632.39 7(a) 6173.25 7(b) 399 472.68
7(c) 6464.16 8(a) 8069.63 8(b) 392 058.59 8(c) 6313.79 9(a) 10 402.15
9(b) 31 9(c) 282.36 9(e) 284.59 10(a) 15 908.46 10(b) 54 ; 14 446.23
11(a) 5306.66 11(b) 371 864.91 11(c) 4638.10 12. 26 550.49 13(a) 1962.60 13(b) 6361.86
CHAPTER 4
REVISION EXERCISE
1. 1 5 2(a) − 5 13 2(b) 25 169 2(c) − 5 13 3(a) −1 3(b) − cosθ sinθ 3(c) cos2 θ
3(d) −1 4(a) t 4(b) −t 4(c) −t 4(d) −t 4(e) − 1 1+t2 4(f) t 1+t2
5(a) 1 5(b) 1 5(c) −2 7(a) θ = 33.21° + k.180° ; θ = 146,79° + k.180°
7(b) θ = 109.65° + k.180° ; θ = 289.65° + k.180° 8. x = −41.54°
EXERCISE 1
1(a) cos x cos 40° + sin x sin 40° 1(b) sin x cos 50° − cos x sin 50° 1(c) cos 2x cos y − sin 2x sin y
1(d) sin10°. cos B + cos10°. sin B 1(e) cos A + 3 sin A 2 1(f) 5 2 (cos x − sin x)
EXERCISE 2

2(a) \( \frac{\sqrt{6} - \sqrt{2}}{4} \) 2(b) \( \frac{\sqrt{6} - \sqrt{2}}{4} \) 2(c) \( -\frac{\sqrt{6} - \sqrt{2}}{4} \) 3(a) \( \hat{A} = 180^\circ - (\hat{B} + \hat{C}) \)

EXERCISE 3

1(a) \( \cos^2 x = \sin^2 x \); \( 2 \cos^2 x - 1 = 1 - 2 \sin^2 x \) 1(b) \( 2 \sin 30 \). \( \cos 30 \)
1(c) \( \cos^2 50 - \sin^2 50 \); \( 2 \cos^2 50 - 1 = 1 - 2 \sin^2 50 \) 1(d) \( 2 \sin 25^\circ \). \( \cos 25^\circ \)
1(e) \( \cos^2 10^\circ - \sin^2 10^\circ \); \( 2 \cos^2 10^\circ - 1 = 1 - 2 \sin^2 10^\circ \) 1(f) \( 2 \sin 35^\circ \). \( \cos 35^\circ \)
1(g) \( 8 \sin 22^\circ \). \( \cos 22^\circ \) 1(h) \( 2 \cos^3 11^\circ - 2 \sin^3 11^\circ \); \( 4 \cos^3 11^\circ = 2 \); \( 2 - 4 \sin^3 11^\circ \)
1(i) \( \sin^4 43^\circ - \cos^4 43^\circ \); \( 1 - 2 \cos^2 43^\circ \); \( 2 \sin^2 43^\circ - 1 \) 2(a) \( \sin 2x \) 2(b) \( \sin 6A \) 2(c) \( \sin 40^\circ \)
2(d) \( \sin 70^\circ \) 2(e) \( \sin 20 \) 2(f) \( \cos 60 \) 2(g) \( \cos 40 \) 2(h) \( \cos \alpha \) 2(i) \( \cos 84^\circ \) 2(j) \( \cos 44^\circ \)
2(k) \( \cos 40 \) 2(l) \( \frac{1}{2} \sin 20 \) 3(a) \( \frac{\sqrt{3}}{2} \) 3(b) \( \frac{1}{2} \) 3(c) \( -\frac{1}{2} \) 3(d) \( -1 \) 3(e) \( -\frac{1}{4} \)
3(f) \( \frac{\sqrt{3}}{2} \) 3(g) \( 1 \) 3(h) \( \frac{1}{2} \) 3(i) \( \frac{\sqrt{3}}{2} \) 3(j) \( -\frac{\sqrt{3}}{2} \) 3(k) \( -\frac{\sqrt{2}}{2} \) 3(l) \( \frac{\sqrt{3}}{2} \)

EXERCISE 4

1(a) \( \frac{1}{2} \) 1(b) \( \frac{2}{\sqrt{2}} \) 1(c) \( -\frac{1}{2} \) 1(d) \( \frac{1}{2} \) 1(e) \( 0 \) 1(f) \( 1 \) 1(g) \( \frac{\sqrt{3}}{2} \)

1(h) \( 4 \) 1(i) \( \frac{\sqrt{3} + \sqrt{2}}{4} \) 2 \( , \) 2 \( , \) 3 \( , \) 1 \( . \) 4 \( . \) \( 4 \cos^3 A - 3 \cos A \)

EXERCISE 5

1(a) \( \frac{120}{169} \) 1(b) \( \frac{119}{169} \) 2(a) \( -\frac{4\sqrt{5}}{9} \) 2(b) \( \frac{1}{9} \) 2(c) \( -4\sqrt{5} \) 3 \( \frac{21}{221} \) 4(a) \( 1 \) 4(b) \( 0 \)

5(a) \( -\frac{2}{3} \) 5(b) \( -\frac{\sqrt{5}}{2} \) 5(c) \( \frac{\sqrt{5}}{6} \) 6(a) \( k \) 6(b) \( 2k^2 - 1 \) 6(c) \( \sqrt{1 - k^2} \)

6(d) \( 2k\sqrt{1 - k^2} \) 7(a) \( 1 - 2k^2 \) 7(b) \( 2k\sqrt{1 - k^2} \) 7(c) \( \sqrt{1 - k^2} \) 7(d) \( \frac{\sqrt{1 - k^2} + 3k}{2} \)

8. \( 2k\sqrt{1 - k^2} \) 9. \( \frac{p + \sqrt{3 - 3p^2}}{2} \)

EXERCISE 6

2(b) \( \frac{1}{2 + \sqrt{2}} \) 3(b) \( \frac{1}{2 - \sqrt{3}} \)

EXERCISE 7

1(a) \( 0^\circ = 55^\circ + k \cdot 90^\circ \) 1(b) \( 0^\circ = 0^\circ + k \cdot 360^\circ ; 180^\circ + k \cdot 360^\circ ; 60^\circ + k \cdot 360^\circ \); \( 300^\circ + k \cdot 360^\circ \)
1(c) \( 0^\circ = 30^\circ + k \cdot 360^\circ ; 150^\circ + k \cdot 360^\circ ; 90^\circ + k \cdot 360^\circ \)
1(d) \( 0^\circ = 0^\circ + k \cdot 360^\circ ; 180^\circ + k \cdot 360^\circ ; 210^\circ + k \cdot 360^\circ ; 330^\circ + k \cdot 360^\circ \)
1(e) \( 0^\circ = 161,57^\circ + k \cdot 180^\circ ; 341,57^\circ + k \cdot 180^\circ ; 135^\circ + k \cdot 180^\circ ; 315^\circ + k \cdot 180^\circ \)
1(f) \( 0^\circ = 90^\circ + k \cdot 360^\circ ; 270^\circ + k \cdot 360^\circ ; 0^\circ + k \cdot 360^\circ ; 360^\circ + k \cdot 360^\circ \) 1(g) \( 0^\circ = 45^\circ + k \cdot 180^\circ \)
2(b) \( 0^\circ \in \{135^\circ ; 315^\circ \} \) 3(a) \( \beta \in \{-90^\circ ; 30^\circ ; 90^\circ ; 150^\circ \} \) 3(b) \( \beta \in \{30^\circ ; 150^\circ \} \)
3(c) \( \beta \in \{0^\circ ; 180^\circ \} \) 4(b) \( \beta = 120^\circ + k \cdot 360^\circ \); \( \beta = 240^\circ + k \cdot 360^\circ \) 5. \( \beta \in \{-150^\circ ; 30^\circ \} \)
6. \( \theta = 71,6^\circ + k \cdot 180^\circ ; 288,4^\circ + k \cdot 180^\circ \)

EXERCISE 8

(a) \( \theta = -8,3^\circ + k \cdot 120^\circ ; 295^\circ + k \cdot 360^\circ \) (b) \( \theta = 10^\circ + k \cdot 120^\circ ; 34^\circ + k \cdot 72^\circ \)
(c) \( \theta = 20^\circ + k \cdot 120^\circ ; -240^\circ - k \cdot 360^\circ \) (d) \( \theta = 42,5^\circ + k \cdot 180^\circ \)
(e) \( \theta = 110^\circ + k \cdot 120^\circ ; -330^\circ - k \cdot 360^\circ \) (f) \( \theta = 60^\circ + k \cdot 60^\circ ; 120^\circ + k \cdot 60^\circ \)
EXERCISE 9
(a) $x = 45^\circ + k \cdot 180^\circ$; $x = 225^\circ + k \cdot 180^\circ$
(b) $x = 45^\circ + k \cdot 180^\circ$; $x = 225^\circ + k \cdot 180^\circ$
(c) $\theta = 120^\circ + k \cdot 360^\circ$; $240^\circ + k \cdot 360^\circ$; $90^\circ + k \cdot 180^\circ$
(d) $A = 45^\circ + k \cdot 90^\circ$; $A = 90^\circ + k \cdot 180^\circ$
(e) $A = 90^\circ + k \cdot 180^\circ$; $A = k \cdot 180^\circ$
(f) $\theta = 90^\circ + k \cdot 180^\circ$; $30^\circ + k \cdot 60^\circ$
(g) $\alpha = 90^\circ + k \cdot 180^\circ$

EXERCISE 10

1(a) [Graph]
1(b) [Graph]
1(c) [Graph]
1(d) [Graph]
1(e) [Graph]
1(f) [Graph]
1(g) [Graph]

2(a) [Graph]
2(b) [Graph]
2(c) [Graph]
2(d) [Graph]

EXERCISE 11
1(a) $180^\circ$
1(b) $x \in \{-30^\circ; 90^\circ\}$
1(d)(1) $-30^\circ < x < 90^\circ$
1(d)(2) $-90^\circ \leq x < -30^\circ$
1(d)(3) $-90^\circ \leq x < 45^\circ$ or $45^\circ < x \leq 90^\circ$
1(d)(4) $0^\circ \leq x \leq 90^\circ$
1(d)(5) $-90^\circ \leq x < 0^\circ$
1(e) [Graph]
EXERCISE 12

1(a) PR = 30.7 cm
1(b) RT = 29.35 cm
1(c) RPT = 57.11°
2(a) 207.64 m
2(b) 90°
2(c) ADB = 34.42°
3(a) AC = 15 m
3(b) 29.03°
3(c) DAC = 36.87°
4(a) AC = 338.83 m
4(b) θ = 7.39°
5(a) 67.88°
5(b) 78.74 m²
6(a) HG = 12.5 m
6(b) GJ = 32.77 m
7(a) ₋ = 21.8°
8(a) 35√2 mm
8(b) θ = 45°

EXERCISE 13

1(a) QS = x sin α / sin β
1(b) PS = x sin α tan θ / sin β
2(c) 119.18 units
3(a) ₋ = 180° - 2θ
3(c) ABD = 60°

REVISION EXERCISE

1(a) -8/17
1(b) -161/289
1(c) 161/240
2(a) -8/17
2(b) 3/√3 34
3(a) -5/4
3(b) 5/4

4(a) -k
4(b) 2k² - 1
4(c) √3k - √(1-k²)
5. m/√(1-m²)
6(a) 2 sin x
6(b) 1
6(c) -tan 4x
6(d) -1/2
7(a) 1/2
7(b) -√2
7(c) 1
7(d) 2 + √2
8(b) x ∈ (-135°; 45°)
9(a) θ = 62.3° + k.120°; θ = 117.7° + k.120°
9(b) θ = 25.32° + k.60°; θ = 85.32° + k.60°
9(c) θ = 146.31° + k.180°; θ = 326.31° + k.180°
10(a) θ = 30° + k.120°; θ = 90° + k.360°
10(b) θ = 67.5° + k.90°
10(c) θ = 90° + k.360°; 270° + k.360°
10(d) θ = 31.72° + k.180°; 121.72° + k.180°
10(e) θ = 270° + k.360°
11. θ ∈ (-116.57°; -63.43°; 63.43°; 116.57°)
12. x ∈ (-180°; -26.57°; 153.43°; 180°)
13(b) 2 cos 3x sin 2x
13(c) x = 30° + k.120°; x = 90° + k.120°; x = 0° + k.180°; x = 90° + k.180°

SOME CHALLENGES

1. 2p√1 - p²
2. P = √3 sin x
3. m = -1/8
5. tan θ = -4/5
6. 14:00 ; 22:00
7(b) 8
7(c) x = 45°
8(b) 16-15√5 / 51
CHAPTER 5

REVISION EXERCISE

(a) \((x - 2)(x^2 + 2x + 4)\)  
(b) \((2x + 1)(4x^2 - 2x + 1)\)  
(c) \((x + 1)(x^2 - 2)\)  
(d) \(27(x + 1)(x^2 - x + 1)\)  
(e) \((x - 3)(x^2 - 3)\)  
(f) \((x + 3)(x^2 + 2)\)  
(g) \(- (x - 1)(x - 3)(x + 3)\)  
(h) \((4x - 1)(x - 2)(x + 2)\)

EXERCISE 1

1(a) 17  
1(b) -1  
1(c) 0  
1(d) -4  
2. \((x + 2)(4x - 9) + 23\)  
3. \((2x - 1)(x + 4) - 1\)

EXERCISE 2

1(a) 17  
1(b) -1  
1(c) 0  
1(d) 23  
1(e) -4  
2.\(-32\)  
3. \(a = -4\)

EXERCISE 3

1(a) \(a = 1, b = -4, c = -4\)  
1(b) \(a = 1, b = 0, c = -1\)  
1(c) \(a = 1, b = 1, c = -3\)

EXERCISE 4

1(a) \(x = 3 \text{ or } -2 \text{ or } -\frac{1}{2}\)  
1(b) \(x = 0 \text{ or } -\frac{2}{3} \text{ or } 2\)  
1(c) \(x = 3 \text{ or } -\frac{1}{3}\)

EXERCISE 5

1(a) \(-1\)  
2. \(R = 3\)  
3. \(x = 3\)  
4. \(-(x + 2)(x^2 - 2x - 1)\)  
5. \(m = 2 \text{ or } m = -4\)  
6. \(q = 1, r = -4\)  
7(a) \((x - 3)(x - 3)\)  
7(b) \((x + 1)(x + 3)\)  
7(c) \((2x - 3)(x^2 - 2)\)

EXERCISE 6

1. \(k = 0 \text{ or } -6 \text{ or } 6\)  
2. \(p = 3, q = -5\)  
3. \(a = 3 \text{ or } -2\)  
4. \(\text{Rem} = 9x\)  
5(a) \(g(6) = -14\)

EXERCISE 1

(a) \(\frac{9}{2}\)  
(b) \(63\)  
(c) \(\frac{1}{2}\)  
(d) \(-\frac{1}{2}\)

EXERCISE 2

1(a) 5  
1(b) -4  
1(c) 4  
2. \(-7\)

EXERCISE 3

1(a) 1  
1(b) 2  
1(c) -3  
1(d) 0  
1(e) 0  
1(f) 0  
1(g) 2x
EXERCISE 4

1(a) 6x^2 1(b) 32x^3 1(c) -35x^4 1(d) \frac{2x^7}{7} 1(e) 0 1(f) -\frac{7}{\frac{1}{3} + \frac{1}{x^2}}
1(g) \frac{\frac{4}{x^6}}{x} 1(h) 0 1(i) 0 1(k) \frac{1}{5} 1(l) \frac{2}{5} x^2
1(m) -\frac{10}{x^3} 1(n) \frac{4}{5x^4}
1(o) 3x^4 + \frac{3}{7} 1(p) 5x^3 1(q) \frac{4}{x^6} 1(r) 13x^2 - 30x + 2
1(s) -1 + 12x^3

2(a) 2x^5 - 6x^7 - \frac{1}{x}
2(b) 3a^2 x^2 + 3x^2 2(c) \pi^3 + 3\pi x^2

EXERCISE 5

1(a) (-4; 0) (2; 0) (0; 8) 1(b) (-1; 9) 1(c) stygend: x < -1; dalend: x > -1
1(d)

2(a) (3; 0) (-2; 0) (0; -6) 2(b) \frac{1}{2} - \frac{1}{4} 2(c)
2(d) -3 2(f) (4; 6)

EXERCISE 6

1(a)

1(b)

1(c)

1(f)
EXERCISE 7
1(a) \( f(x) = -x^3 + 5x^2 + 8x - 12 \) \quad 1(b) \( f(x) = -x^3 + x^2 + 8x - 12 \) \quad 2(a) \( a = 2 \); \( b = 3 \)
2(b) \( a = 1 \); \( b = -3 \) \quad 2(c) \( a = 12 \); \( b = -36 \) \quad 2(d) \( m = 2 \); \( n = 17 \) \quad 2(e) \( b = \frac{1}{2} \); \( c = -10 \frac{1}{6} \)
2(f) \( a = 2 \); \( b = 6 \) \quad 2(g) \( a = -\frac{9}{4} \); \( b = -6 \) \quad 2(h)(1) \( a = 2 \); \( b = 3 \) \quad 2(h)(2) \( (0 ; 0) \) \( (-1 ; 1) \)

EXERCISE 8
1(a) \( f'(x) = -3x^2 + 12x + 15 \); \( f(x) = -x^3 + 6x^2 + 15x \) \quad 1(b)(1) \( -1 < x < 5 \) \quad 1(b)(2) \( x < -1 \) or \( x > 5 \)
1(c) \( x = -1 \) or \( x = 5 \) \quad 1(b) \( 2 \) \quad 1(c) \( x < -1 \) or \( x > 5 \) \quad 2(d) \( \text{max: } x = 5 \); \( \text{min: } x = -1 \)

EXERCISE 9
1(a) \( y = -2x + 1 \) \quad 1(b) \( y = -x + 5 \) \quad 2(a) \( y = -26x - 46 \) \quad 2(b) \( y = -\frac{23}{13} \)
3. \( y = -2x + 13 \); \( (7 ; -1) \) \quad 4(a) \( y = -9x + 18 \) \quad 4(b) \( y = \frac{1}{3}x + 3 \) \quad 4(c) \( y = 2x + 1 \)
4(d) \( y = 7x - 7 \) \quad 5(a) \( y = -2x - 14 \) \quad 5(b) \( y = 5x - 20 \); \( y = -5x - 5 \)
6(a) \( (3 ; 9) \) \quad 6(b) \( (4 ; 8) \) \quad 6(c) \( (1 ; 4) \); \( (-1 ; -4) \) \quad 6(d) \( (\frac{1}{2}; -\frac{22}{3}) \)
7(a) \( y = x + 1 \) \quad 7(b) \( p = -18 ; 14 \) \quad 7(c) \( y = 10x - 19 \); \( y = -2x + 5 \)
7(e) \( y = 14x + 53 \); \( y = -2x + 5 \) \quad 7(f) \( (\frac{1}{6} ; \frac{1}{12}) \) \quad 8. \( y = -4x + 18 \)

EXERCISE 10
1. \( 50m ; 50m \) \quad 2(a) \( L = 2y + x \) \quad 2(c) \( 200m \) \quad 3(a) \( BC = x + 30 \); \( AB = \frac{5400}{x} + 20 \)
3(b) \( BC = 120mm \); \( AB = 80mm \) \quad 4(a) \( C = 15 \left( 2x + \frac{648}{x} \right) \) \quad 4(b) \( 18m ; 24m \)
5(b) \( \frac{10000}{\pi} \) \quad 6. \( R = r = 100mm \) \quad 7(e) \( \frac{144}{\pi \times 27} \) \quad m

EXERCISE 11
1(b) \( 10mm \) \quad 2(a) \( h = \frac{324}{x} \) \quad 2(b) \( A = 6x^2 + \frac{592}{x} \) \quad 2(c) \( 18cm ; 6cm ; 9cm \)
3(b) \( V = 120x - \frac{18x^3}{3} \) \quad 3(c) \( \frac{20}{3} \) \; \( \text{cm} \); \( 10cm ; 4cm \) \quad 4(a) \( A = x^2 + 4xh \) \quad 4(c) \( C = 10x^2 + \frac{1280}{x} \)
4(d) \( x = 4m \); \( h = 2m \) \quad 5. \( r = 11.7cm \); \( h = 23.4cm \) \quad 6(a) \( h = \frac{462-2x^3}{2x} \) \quad 6(b) \( r = 7cm \)
7(a) \( r = 12 - h \) \quad 7(b) \( V = 48xh - 8xh^2 + \frac{2}{3}h^3 \) \quad 7(c) \( \frac{256}{3} \pi \)

EXERCISE 12
1. \( 2 \) \quad 2. \( 2 \) \quad 3(a) \( A(x; 18 - 2x^2) \); \( B(-x; 18 - 2x^2) \) \quad 3(b) \( 36x - 4x^3 \)
3(c) \( 24\sqrt{3} \) \quad 4(a) \( PQ = 2x \); \( QR = -\frac{1}{2}x^2 + 6 \) \quad 4(b) \( 16 \) \quad 5(a) \( OA = 8 \); \( OB = 2 \)

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5(b) $OC = 8 - x^3$
5(d) $\frac{2}{\sqrt{y}}$

EXERCISE 13
1(a) 50 1(b) 24 1(c) 36 2. $10\,000\pi$
3(a) 24 3(b) $t = 1$; $t = 9$
3(c) $10 - 2t$
3(d) Increasing at $t = 4$; Stationary at $t = 5$; Decreasing at $t = 6$
3(e) 5 3(f) 25 3(g) $t > 5$
3(h) $t = 0$; $t = 10$ 3(i) $t = 2$ 3(j) $t = 9$

EXERCISE 14
1(a) $18m$
1(b) 5 1(c) $50m$
1(d) $t = 2$; $t = 8$
1(e) $68m$
1(f) $t = 10$
1(g) $0 < t < 5$
1(h) $5 < t < 10$
1(i) 20
1(j) Velocity increasing
1(k) 4
1(l) Not moving
1(m) Velocity decreasing
1(n) 9
1(o) -4
2(a) $34m$
2(b) 5
2(c) $32.25m$
2(d) $t = 3$

REVISION EXERCISE
1(a) $-1$
1(b) 2
1(c) $y = 2x + 4$
2(a) $\frac{1}{\sqrt{x}} - \frac{8}{x^2}$
2(b) $\frac{12}{x^3} + \frac{1}{x^7}$
2(c) $\frac{5}{2}x^{\frac{1}{2}} - 3x^2 + \frac{3}{2x^2}$
2(d) $\frac{-18}{x^3} - \frac{12}{x^{\frac{1}{2}}} + 16$
2(e) 8y
2(f) $\frac{2a^2}{-3t^4}$
3. $k = -4$
4(a) $-\frac{1}{x^2}$
4(b) $-6x^2$
4(c) 5
5. $y = 7x - 11$

6.

8(a)

8(b) $k > 5$ or $k < -7$
9. $a = 21; b = -60; c = 43$
10. $k = 0$; $k = 4$
11(a) $a = 1; b = -9$
11(b) $(0; -20)$
11(c) 3
11(d) $(4; -4)$
11(e) $(2; -2)$
12(a) $x > -1$
12(b) $x = -1$

13(a) $(0; -4)$
13(b) $y = x^2 - 4x + 3$
13(c) $x = 1$; $3$
13(d) $x < -1$ or $x > 3$
13(e) $1 < x < 3$
13(f) $x = 2$
13(g)

14(a)(1) $P = 2h + 2r + \pi r$
14(a)(2) $A = 2rh + \frac{1}{2}\pi r^2$
14(c) $1.06m$
15(a) $V = \left(6 - x\right)\left(4 + 2x\right)\left(3\right)$
15(b) $x = 2$
15(c) $96m^3$
16(a) $1000$ million
16(b) $20$ million per hour
16(c) $t > 5$
16(d) $20$ hrs

17(a) $-0.6$ metres per hour
17(b) Decreases at $\frac{9}{4}$ metres per hour

18(a) $35m$
18(b) $30$ metres per second
18(c) $80$ metres

18(d) $20$ metres per second
18(e) $40$ metres per second

SOME CHALLENGES
1(a) $\frac{a^2}{x^2}$
1(b) $\frac{1}{2}x^2 - 21$
1(c) $f(x) = \frac{1}{3}x^2 + 4x + 5$
1(d) $3\frac{1}{3}$
1(e) $\frac{1}{2\sqrt{x}} - \frac{5}{x^2}$
1(f) 16
1(g) $\frac{3}{5}$ hrs
2(a) $24km$
3. $(4 \frac{1}{2}; 1)$
4(b) $x = \frac{2}{4}$

5(b) $A(x) = 400x - 2x^2$
5(c) $100mm; 200mm$

CHAPTER 7

REVISION EXERCISE
1(a) $G\left(1; -4\right)$
1(b) $CG = \sqrt{26}$
1(c) $\theta = 103,43^\circ$
1(d) $y = -3x - 9$
1(e) $y = \frac{1}{3}x - \frac{7}{3}$
1(f) $-\frac{7}{3}$
2(a) $k = 4$
2(b) $k = -\frac{1}{2}$
2(c) $k = -13$
2(d) $k = 4$
EXERCISE 1
1(a) $x^2 + y^2 = 13$  1(b) $x^2 + y^2 = 11$  1(c) $x^2 + y^2 = 37$  1(d) $x^2 + y^2 = 9$

2(a) $y = 2(b) y = 2(c) y = 2(d) y =$

3(a) $x^2 + y^2 = 16$  3(b) $x^2 + y^2 = 10$  3(c) $x^2 + y^2 = 20$

EXERCISE 3
1(a) $3(x: -1)$  1(b) $m = 8$  1(c) $A(-4; -2), B(2; 4)$  1(d) $AB = \sqrt{72}$

EXERCISE 2
1(a) $y = \frac{3}{2}x - \frac{13}{2}$

EXERCISE 3
2(a) $x = 2$ and $x = 6$  2(b) $x = 1$ and $y = 7$

REVISION EXERCISE
1(a) $(x+1)^2 + (y-4)^2 = 25$  1(b) $Q(-5; 7)$  1(d) $m_{PM} \times m_{MQ} = -1 \times 1 = -1$

REVISION EXERCISE
2. $(x+1)^2 + (y-4)^2 = 8$

SOME CHALLENGES
1. Area = 1 units$^2$  2. $k = 9$  4. $(x+1)^2 + (y-4)^2 = 25$  5. $F(15; 5)$  6. $F(2+3\sqrt{2}; 5)$

CHAPTER 8

REVISION EXERCISE
1. 32cm  2(a) 40°  2(b) 40°  2(c) 75°  2(d) 15°  4(b) $\hat{B}_1; \hat{Q}_3; \hat{R}_2; \hat{R}_1$
4(c) \(90^\circ - 2x\)

**EXERCISE 1**
1. 4cm  2. 15cm  3. 4 units  4. 10cm  5. \(DG = 20\text{mm}, GE = 12\text{mm}; DH = 15\text{mm}, HF = 9\text{mm}\)
6. 15mm  9(a) 1:1  9(b) 1:1  9(c) 1:2  9(d) 1:2  10. 2:1
12. 24mm  13(a) 9mm  13(b) 10mm  14(a) RA:RP = 2:3  RB:BQ = 1:2  14(b) 1:2
15(c) 4cm  16(a) 8:15  16(b) 4:5  17(a) 3:2  17(b) 3:7  17(c) 2:5  17(d) 25:49

**EXERCISE 3**
2. 45cm  3. 24 units  8(a) 1:1  8(b) 1:1  9(c) 1:2  9(d) 1:2
10. 2:1

**EXERCISE 4**
5(b)(1) 7cm  5(b)(2) 6.72cm

**EXERCISE 5**
9(b) 4 units

**EXERCISE 8**
5(a) 1 unit  5(b) \(2\sqrt{3}\) units  6(b)(1) \(\frac{1}{2}\) BC  6(b)(2) \(\frac{3}{2}\) BC

**REVISION EXERCISE**
1(a) 2.3  1(b) 4  2(a) 6\(\sqrt{2}\)  2(b) 2\(\sqrt{2}\)  3(b)(1) 10
3(b)(2) 22.5  3(b)(3) 12.5  4(a) 2  4(b) 2  4(c) \(\frac{4}{5}\)
8(a) 10  8(b) 14  9(b) 48cm ; 73cm  9(c) 105.6cm

**SOME CHALLENGES**
4(b) 2  5(b) 2

**CHAPTER 9**

**REVISION EXERCISE**
1(a) 4,6  1(b) 5  1(c) 1(d) Positively skewed  1(e) 1.8  1(f) 3 athletes  1(g) Falls outside the interval 
\((3-1.5 \times 2 ; 5+1.5 \times 2) = (0 ; 8)\)

1(h)

2(a) 10%  2(b) 2(c) Positively skewed  2(d) Khaya and Tsidi

2(e)

2(f) 9,9  2(g) Khaya and Tsidi

3(a) Same median and max value  3(b) No  3(c) Negatively skewed  3(d) min value 32  3(e) No outliers for B.
4(b) 34,4  4(c) 30 < x \leq 40  4(e) \(Q_1 = 26, M = 34, Q_3 = 43\)
4(f) 36km

4(g)

4(h) Positively skewed
5(a) 80  5(b) 38  5(c) 42  5(d) 58-60km/u  5(e) 62  5(f) 8,75%  5(h) 63,4%  5(i) 22,4

**EXERCISE 1**
1(b) Linear function  1(c) \(y = 0.8571428571x - 17.14285714\)  1(e)(1) 111,4285714
1(e)(2) 94,28571428  2(b) \(y = 2.41776127 + 0.0742191457x\)  2(d) 4.3
3(a)(2) As the temperature increases, the number of flowers increases.  
3(a)(3) \( y = -26.6 + 1.14x \)  
3(a)(4) 17 flowers  
3(b)(2) \( y = 118.1 - 2.7x \)  
3(b)(4) 44°C  
4(b) Quadratic  
5(b) 0.065  
5(c) \( y = 163.4268962 + 6375.623752x \)  
5(e) 801  
5(f) Yes due to high accidents at 0.05  
6(a) Quadratic  
7(a) 55-64 year olds  
7(b) Underweight category  
7(c) Quadratic trend for all four.  

**EXERCISE 2**  
1. \( r = 0.5 \)  
2. \( r = -0.98 \)  
3. \( r = 0.947 \)  
4. \( r = -0.995 \)  
5. \( r = -0.3 \)  
6(a) \( r = 0.99 \)  
6(b) \( r = 0.967 \)  
6(c) \( r = 0.969 \)  
6(d) \( r = -0.925 \)  
6(e) \( r = 0.964 \)  

**REVISION EXERCISE**  
1(b) \( y = 118.6 + 28.4x \)  
1(d) 913  
1(e) 771  
1(f) 999  
1(g) \( r = 0.988 \)  

**CHAPTER 10**  

**REVISION EXERCISE**  
1(a) A, D, G  
1(b) B, D, H  
1(c) A, C, G  
1(d) B, C, E, F, H  
2(a) \( S = \{1 ; 2 ; 3 ; 4 ; 5 ; 6\} \)  
2(b) \( P(A) = \frac{1}{6} \)  
2(c) \( \frac{1}{2} \)  
2(d) mutually exclusive  
2(e) \( \frac{1}{2} \)  
2(f) \( \frac{2}{3} \)  
2(g) No  
3. 70%  
4. 0.67  
5(a) \( \frac{4}{5} \)  
5(b) \( \frac{13}{20} \)  
5(c) \( \frac{6}{11} \)  
5(d) \( \frac{225}{5247} \)  
5(e) \( \frac{7}{5} \)  
5(f) \( \frac{1}{2} \)  
5(g) No  
6(a) \( \frac{460}{5247} \)  
6(b) \( \frac{478}{5247} \)  
7(b) 0.032  
7(c) 0.0665  
8(a)  
8(b) \( x = 11 \) and 11 play rugby and cricket  
8(c)(1) \( \frac{27}{240} \)  
8(c)(2) \( \frac{29}{48} \)  
8(c)(3) \( \frac{1}{60} \)  

**EXERCISE 1**  
1. \( 4 \times 5 \times 3 = 60 \)  
2(a) \( 7 \times 7 \times 7 \times 7 \times 7 = 7^7 = 823543 \)  
2(b) \( 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040 \)  
2(c) \( 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040 \)  
3(a) \( 6 \times 6 \times 6 \times 6 \times 6 \times 6 = 6^6 = 46656 \)  
3(b) \( 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720 \)  
4. \( 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64 \)  
5(a) \( 10 \times 10 \times 10 \times 26 \times 26 = 676000 \)  
5(b) \( 109 \times 89 \times 26 \times 25 = 468000 \)  
5(c) \( 9 \times 9 \times 9 \times 21 \times 21 = 321489 \)  
5(d) \( 9 \times 8 \times 7 \times 21 \times 20 = 211680 \)  
5(e) \( 4 \times 4 \times 4 \times 16 \times 16 = 16384 \)  
6(a) \( 4 \times 3 = 12 \)  
6(b) \( 4 \times 3 \times 3 \times 3 = 44 \)  
6(c) \( 4 \times 3 \times 2 \times 3 = 72 \)  

**EXERCISE 2**  
1(a) 19958400  
1(b) 360360  
2. \( 720 \)  
3. \( 116 \times 280 \)  
4(a) 823543  
4(b) 5040  
4(c) 2401  
4(d) 840  
5(a) 16777216  
5(b) 40320  
5(c) 262144  
5(d) 20160  
6(a) 6561  
6(b) 3024  
6(c) 2025  
6(d) 2024  

**EXERCISE 3**  
1(a) 9!  
1(b) 20!  
2(a) 5!  
2(b) 48  
3(a) 9!  
3(b) 80640  
3(c) 5760  
3(d) 17280  
3(e) 14400  
3(f) 2880  
4(a) 5040  
4(b) 288  
4(c) 576  
5(a) 12!  
5(b) 41472  
5(c) 360  
7. \( 165888 \)  
8(a) 12!  
8(b) 79833600  
8(c)(1) 16  
8(c)(2) 12  

**EXERCISE 4**  
1(a) 5040  
1(b) 2520  
1(c) 120  
1(d) 60  
2(a) 5040  
2(b) 420  
2(c) 80  
2(d) 10  
2(e) 120  
2(f) 180  
3(a) 10!  
3(b) 1814400  
3(c) 40320  
3(d) 362880  
3(e) 181440  
4(a) 8!  
4(b) 56  
5(a) 6  
5(b) 15  
5(c) 20  

345
EXERCISE 5

1(a) \( \frac{1}{30} \) 1(b) \( \frac{1}{6} \) 2(a) \( \frac{1}{72} \) 2(b) \( \frac{1}{9} \)
3(a) \( \frac{1}{189} \) 3(b) \( \frac{40}{3087} \) 3(c) \( \frac{217}{729} \) 4. \( \frac{14}{45} \)
5(a) \( \frac{1}{84} \) 5(b) \( \frac{1}{7} \) 6(a) \( \frac{1}{504} \) 6(b) \( \frac{1}{36} \)
7(a) \( \frac{1}{624} \) 7(b) \( \frac{1}{624} \) 8(a) \( \frac{11550}{11550} \) 8(b) \( \frac{1}{27} \)
9(a) \( \frac{1}{72072} \) 9(b) \( \frac{1}{12012} \) 9(c) \( \frac{2}{13} \) 9(d) \( \frac{2}{439} \)

REVISION EXERCISE

1(a) 24 1(b) 8 2. \( \frac{1}{10000} \) 3(a) \( 9^9 \)
3(b) \( 9! \) 3(c) \( 9^2 \) 3(d) 3024 3(e) 5040
3(f) \( \frac{1}{72} \) 4(a) \( 13! \) 4(b) \( \frac{13!}{35 \times 2!} \) 4(c) \( \frac{14!}{2!} + \frac{15!}{3!} \)
4(d) \( \frac{2}{39} \) 4(e) \( \frac{12!}{35 \times 2!} \) 4(f) \( \frac{1}{13} \)
5(a) \( 8000000 \) 5(b) \( 16000000 \) 6(a) 4096 6(b) 1680
6(c) 1535 6(d) 671 6(e) \( \frac{1695}{4096} \) 6(f) \( \frac{1}{2} \)
6(g) 511 7(a) \( 11! \) 7(b) 241920
7(c) \( 10160640 \) 7(d) \( \frac{14}{55} \) 7(e) 7257600

SOME CHALLENGES

1(a) \( a = 450 ; b = 319 ; c = 268 ; d = 718 \) 1(b) \( \frac{183}{1470} \) 1(c) \( \frac{268}{1470} \) 1(d) \( \frac{450}{718} \)
1(e) yes 2. \( \frac{2}{5} \times \frac{5}{9} + \frac{1}{3} \times \frac{2}{5} = \frac{58}{135} \) 3. \( \frac{1}{10} \times \frac{3}{10} + \frac{7}{10} \times \frac{7}{10} = \frac{54}{100} = \frac{27}{50} \)
4. \( 3 \times 3! \times 4! = 432 \) 5(a) 40 5(b) 160 5(c) 160 6. \( \frac{7}{21 \times 3! \times 2!} = 210 \)